Jump-Diffusion with Stochastic Volatility and Intensity

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Abstract: An alternative option pricing model is proposed, in which the asset prices follow the jump-diffusion with stochastic volatility and intensity. The stochastic volatility follows the jump-diffusion. We find a formulation for the European-style option in terms of characteristic functions.

Keywords: Jump-diffusion model, Stochastic Volatility, Intensity, Characteristic functions.

1. Introduction

In 1973, Fischer Black and Myron Scholes introduced a theoretical valuation formula for options is derived. In 1993, Heston studied a new technique to derive a closed – form solution for the price of a European call option on an asset with stochastic volatility. The Heston model assumes that $S_t$, the price of the asset, is determined by a stochastic process:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW^S_t$$ (1)

where $\mu > 0$, $v_t$ the instantaneous variance is a CIR process:

$$dv_t = \kappa_v (\theta_v - v_t) dt + \sigma \sqrt{v_t} dW^v_t$$ (2)

and $\kappa_v > 0, \theta_v > 0, \sigma > 0$, $W^S_t, W^v_t$ are Brownian motion with correlation $\rho$.

In 1996, Bates introduced an efficient method is developed for pricing American options on stochastic volatility /jump-diffusion processes under systematic jump and volatility risk. The exchange rate $S_t$ satisfy the following process:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW^S_t + kdN_t$$

$$dv_t = \kappa_v (\theta_v - v_t) dt + \sigma \sqrt{v_t} dW^v_t$$ (3)

where $k$ is the random percentage jump conditional on a jump occurring and $N_t$ is a Poisson process with constant intensity $\lambda$. 
2. Model Descriptions

The propose model assumes that the underlying asset has the following dynamics under risk-neutral measure,

\[
\frac{dS_t}{S_t} = (r - \lambda_t) dt + \sqrt{v_t} dW^S_t + Y_t dN_t,
\]

\[
dv_t = \kappa_v (\theta_v - v_t) dt + \sigma_v \sqrt{v_t} dW^v_t,
\]

\[
d\lambda_t = \kappa_\lambda (\theta_\lambda - \lambda_t) dt + \epsilon_\lambda \sqrt{v_t} dW^\lambda_t
\]

where \( S_t, \, v_t, \, \kappa_v, \, \theta_v, \, \sigma_v, \, Y_t, \, N_t, \, W^S_t \) and \( W^v_t \) are define (1), (2) and (3). \( r \) is the risk-free rate, \( m \) is the expected of \( Y_t \), \( \kappa_\lambda \) is a mean-reverting rate. We assume that jump process \( N_t \) are independent of \( W^S_t, \, W^v_t \) and \( W^\lambda_t \). A standard Brownian motion \( W^S_t, W^v_t \) and \( W^\lambda_t \) are independent.

3. Characteristic Functions

Denote the characteristic function as

\[
f(l, v, \lambda, t; x) = E[e^{ixY_t} \mid X_t = l, v_t = v] \tag{5}
\]

where \( T \geq t \) and \( i = \sqrt{-1} \). Then, the following theorem holds.

**Theorem 3.1** Suppose that \( S_t \) follows the dynamics in (4). Then the characteristic function for \( X_T \) defined in (5) is given by

\[
f(l, v, \lambda, t; x) = \exp(ixl + i\lambda r_t + A(\tau) + B(\tau)v + C(\tau)\lambda),
\]

where

\[
A(\tau) = -2\kappa_\theta \frac{\theta_v}{\sigma_v^2} \ln \left[ \frac{r_v e^{-\frac{1}{2}r_v^2} + r_v e^{\frac{1}{2}r_v^2}}{2H} \right], \quad B(\tau) = \left( u^2 - u \right) \left( \frac{1 - e^{-Ht}}{r_v + r_v e^{-Ht}} \right), \quad C(\tau) = 2F \left[ \frac{1 - e^{-E\tau}}{q_1 + q_2 e^{-E\tau}} \right],
\]

\[
r_v = (\kappa_v - \rho \sigma u) + H, \quad r_v = (\kappa_v - \rho \sigma u) + H, \quad H = \sqrt{(\kappa_v - \rho \sigma u)^2 - \sigma_v^2 (u^2 - u)}\]

\[
q_1 = \kappa_\lambda + E, \quad q_2 = -\kappa_\lambda + E, \quad E = \sqrt{\kappa_\lambda^2 - 2\epsilon_\lambda^2 F}, \quad F = -mu + \int_{-\infty}^{\infty} (e^{\rho y} - 1) \phi_\rho(y) dy
\]

and \( \phi_\rho(y) \) is a density of random jump size \( Y_t \).

**Proof** Feynman-Kac formula gives the following PDE for the characteristic function

\[
(r - \frac{1}{2} \sigma^2) f_t + \frac{1}{2} \sigma^2 f_{tt} + \kappa_v (\theta_v - v) f_t + \frac{1}{2} \sigma^2 v_t f_v + \rho \sigma v f_{tv} + \kappa_\lambda (\theta_\lambda - v) f_\lambda + \frac{1}{2} \epsilon_\lambda^2 f_{\lambda\lambda} + \lambda \int_{-\infty}^{\infty} [f(l+y, v, \lambda, t; \phi) - f(l, v, \lambda, t; \phi)] \phi_\rho(y) dy + f_t = 0, \tag{6}
\]
\[ f(l, v, \lambda, T; x) = e^{ixl}. \]

Consider form for the characteristic function:

\[ f(l, v, \lambda, t; x) = \exp(ixl + ixr\tau + A(\tau) + B(\tau)v + C(\tau)\lambda) \]  \hspace{1cm} (7)

where \( \tau = T - t \) and \( A(\tau = 0) = B(\tau = 0) = C(\tau = 0) \).

We plan to substitute equation (7) into equation (6). Firstly, we compute

\[ f_l = ixf, \quad f_v = -x^2f, \quad f_v = B(\tau)f, \quad f_w = B^2(\tau)f, \quad f_{\lambda \lambda} = C^2(\tau)f, \quad f_t = -(ixr - A_\tau - B_v - C_\lambda)f, \]

\[ f(l + y, v, \lambda, t; x) - f(l, v, \lambda, t; x) = e^{iyf}. \]

Substitute all terms above in equation (6),

\[ (r - \frac{1}{2}v - \lambda m)xf + \frac{1}{2}(-x^2f) + \kappa_v(\theta_r - v)B(\tau)f + \frac{1}{2}\sigma^2 vB^2(\tau)f + \rho \sigma vixB(\tau)f \]

\[ + \kappa_\lambda (\theta_\lambda - \lambda)C(\tau)f + \frac{1}{2}e^{2\lambda}C^2(\tau)f + \lambda \int_{-\infty}^{\infty} e^{iy}\phi_i(y)dy - (ixr + A_\tau + B_v + C_\lambda)f = 0. \]

Let \( ix = u \), then

\[ (r - \frac{1}{2}v - \lambda m)u + \frac{1}{2}vu^2 + \kappa_v(\theta_r - v)B(\tau) + \frac{1}{2}\sigma^2 vB^2(\tau) + \rho \sigma vuB(\tau) \]

\[ + \kappa_\lambda (\theta_\lambda - \lambda)C(\tau) + \frac{1}{2}e^{2\lambda}C^2(\tau) + \lambda \int_{-\infty}^{\infty} e^{iy}\phi_i(y)dy - ru - A_\tau - B_v - C_\lambda = 0. \]

We have

\[ A_\tau + B_v + C_\lambda = \kappa_v \theta_r B(\tau) + \kappa_\lambda \theta_\lambda C(\tau) \]

\[ + \left( \frac{1}{2}u^2 - \frac{1}{2}u - \kappa_v B(\tau) + \frac{1}{2}\sigma^2 B^2(\tau) + \rho \sigma uB(\tau) \right) v \]

\[ + \left[ \frac{1}{2}e^{2\lambda}C^2(\tau) - \kappa_\lambda C(\tau) - mu + \int_{-\infty}^{\infty} (e^{iy} - 1)\phi_i(y)dy \right] \lambda. \]

This leads to the following system:

\[ A_\tau = \kappa_v \theta_r B(\tau) + \kappa_\lambda \theta_\lambda C(\tau) \]  \hspace{1cm} (8)

\[ B_v = -\frac{1}{2}(u^2 - (\kappa_v - \rho \sigma u)B(\tau) + \frac{1}{2}\sigma^2 B^2(\tau) \]  \hspace{1cm} (9)

\[ C_\lambda = \frac{1}{2}e^{2\lambda}C^2(\tau) - \kappa_\lambda C(\tau) - mu + \int_{-\infty}^{\infty} (e^{iy} - 1)\phi_i(y)dy. \]  \hspace{1cm} (10)

In the equation (9) become a Ricatti equation. Let

\[ B(\tau) = -\frac{G'(\tau)}{\sigma^2 G(\tau)}, \]

substitute \( B(\tau) \) in equation (9),
\[- \left[ \frac{\sigma^2}{2} G'(\tau) G''(\tau) - \frac{\sigma^2}{2} (G'(\tau))^2 \right] \frac{1}{\sigma^4 G^2(\tau)} = -\frac{1}{2} (u - u^2) + (\kappa_i - \rho \sigma u) \frac{G'(\tau)}{\sigma^2 G^2(\tau)} + \frac{1}{2} \frac{\sigma^2 (G'(\tau))^2}{\sigma^4 G^2(\tau)} \]

Then

\[ \frac{\sigma^2}{2} \frac{G(\tau) G''(\tau)}{\sigma^4 G^2(\tau)} + \frac{1}{2} (u^2 - u) - (\kappa_i - \rho \sigma u) \frac{G'(\tau)}{\sigma^2 G(\tau)} = 0. \]

Multiply by \( \frac{\sigma^2}{2} G(\tau) \),

\[ G''(\tau) + (\kappa_i - \rho \sigma u) G'(\tau) + \frac{\sigma^2}{4} (u^2 - u) G(\tau) = 0. \]

General solution is

\[ G(\tau) = C_1 e^{-\frac{1}{2} r_1 \tau} + C_2 e^{-\frac{1}{2} r_2 \tau} \]

where

\[ r_1 = (\kappa_i - \rho \sigma u) + H, \quad H = \sqrt{(\kappa_i - \rho \sigma u)^2 - \sigma^2 (u^2 - u)} \]

\[ r_2 = -(\kappa_i - \rho \sigma u) + H. \]

Note that \( r_1 + r_2 = 2H, \quad r_1 r_2 = -\sigma^2 (u^2 - u). \)

The boundary condition

\[ G(0) = C_1 + C_2, \quad G'(0) = -\frac{1}{2} r_1 C_1 + \frac{1}{2} r_2 C_2 = 0. \]

We have \( C_1 = \frac{r_1 G(0)}{2H} \) and \( C_2 = \frac{r_2 G(0)}{2H}. \)

Thus

\[ B(\tau) = -\left[ \frac{G'(\tau)}{G(\tau)} \right] = \frac{\frac{1}{2} \left[ r_1 e^{-\frac{1}{2} r_1 \tau} + r_2 e^{-\frac{1}{2} r_2 \tau} \right]}{\frac{1}{2} \left[ r_1 e^{-\frac{1}{2} r_1 \tau} + r_2 e^{-\frac{1}{2} r_2 \tau} \right]} \]

\[ = \frac{1}{\sigma^2} \left[ \frac{r_1 e^{-\frac{1}{2} r_1 \tau} - r_2 e^{-\frac{1}{2} r_2 \tau}}{r_1 e^{-\frac{1}{2} r_1 \tau} + r_2 e^{-\frac{1}{2} r_2 \tau}} \right] \]

\[ = \frac{1}{\sigma^2} \left[ \frac{-\sigma^2 (u^2 - u) e^{-\frac{1}{2} r_1 \tau} + \sigma^2 (u^2 - u) e^{-\frac{1}{2} r_2 \tau}}{r_2 e^{-\frac{1}{2} r_2 \tau} + r_2 e^{-\frac{1}{2} r_2 \tau}} \right]. \]
\[ (u^2 - u) \left[ -e^{\frac{-q_1\tau}{2}} + e^{\frac{q_2\tau}{2}} \right] \]

\[ = (u^2 - u) \left( \frac{1 - e^{-H\tau}}{r_1 + r_2 e^{-H\tau}} \right). \]

Next, consider in equation (10).

\[ C = \frac{1}{2} e^2 C^2(\tau) - \kappa_\lambda C(\tau) - mu + \int_{-\infty}^{\infty} (e^{\phi_y} - 1)\phi_y(y)dy. \]

Let

\[ C(\tau) = -\frac{M'(\tau)}{e^2 M(\tau)}. \]

Similarly in \( B(\tau) \), we have

\[ M(\tau) = \frac{q_1 M(0)}{2E} e^{\frac{-q_1\tau}{2E}} + \frac{q_2 M(0)}{2E} e^{\frac{q_2\tau}{2E}} \]

where \( E = \sqrt{\kappa_\lambda^2 - 2e^2 F} \), \( F = -mu + \int_{-\infty}^{\infty} (e^{\phi_y} - 1)\phi_y(y)dy \), \( q_1 = \kappa_\lambda + E \), \( q_2 = -\kappa_\lambda + E \).

Thus

\[ C(\tau) = \frac{1}{2} \frac{q_1 M(0)}{2E} e^{\frac{-q_1\tau}{2E}} + \frac{1}{2} \frac{q_2 M(0)}{2E} e^{\frac{q_2\tau}{2E}} \]

\[ = \frac{q_1 q_2 e^{\frac{-q_1\tau}{2E}} - q_1 q_2 e^{\frac{q_2\tau}{2E}}}{e^2 (q_2 e^{\frac{-q_1\tau}{2E}} + q_1 e^{\frac{q_2\tau}{2E}})} \]

\[ = \frac{1}{e^2 (2e^2 F)} \left[ \frac{1 - e^{-E\tau}}{q_1 + q_2 e^{-E\tau}} \right] \]

\[ = 2F \left[ \frac{1 - e^{-E\tau}}{q_1 + q_2 e^{-E\tau}} \right]. \]

Consider in equation (8),

\[ A_\tau = \kappa_\lambda \theta B(\tau) + \kappa_\lambda \theta C(\tau). \]

Integrating with respect to \( \tau \),

\[ A(\tau) = \kappa_\lambda \int_0^\tau B(s)ds + \kappa_\lambda \int_0^\tau C(s)ds \]

\[ = -\frac{2\kappa_\lambda \theta}{\sigma^2} \int_0^\tau G'(s)ds - \frac{2\kappa_\lambda \theta}{e^2} \int_0^\tau M'(s)ds. \]
\begin{align*}
&= -\frac{2\kappa \theta}{\sigma^2} \ln G_s(s) \bigg|_{s=0}^{\tau} - \frac{2\kappa \theta}{e^2} \ln M(s) \bigg|_{s=0}^{\tau} \\
&= \frac{2\kappa \theta}{\sigma^2} \ln \frac{G(\tau)}{G(0)} - \frac{2\kappa \theta}{e^2} \ln \frac{M(\tau)}{M(0)} \\
&= -\frac{2\kappa \theta}{\sigma^2} \ln \left[ \frac{r G(0) e^{-\frac{1}{2} \sigma^2 \tau}}{2HG(0)} + \frac{r G(0) e^{\frac{1}{2} \sigma^2 \tau}}{2HG(0)} \right] \quad -\frac{2\kappa \theta}{e^2} \ln \left[ \frac{q_M(0) e^{-\frac{1}{2} \sigma^2 \tau}}{2EM(0)} + \frac{q_M(0) e^{\frac{1}{2} \sigma^2 \tau}}{2EM(0)} \right] \\
A(\tau) &= -\frac{2\kappa \theta}{\sigma^2} \ln \left[ \frac{r e^{-\frac{1}{2} \sigma^2 \tau} + r e^{\frac{1}{2} \sigma^2 \tau}}{2H} \right] -\frac{2\kappa \theta}{e^2} \ln \left[ \frac{q e^{-\frac{1}{2} \sigma^2 \tau} + q e^{\frac{1}{2} \sigma^2 \tau}}{2E} \right].
\end{align*}

The proof is now completed.

4. A Formula for European Option Pricing

Following Carr and Madan (1999), the modified call price \( c_T(k) \) is defined by

\[ c_T(k) = e^{\alpha k} C_T(k) \quad \text{for some constant } \alpha > 0 \]

where \( C_T(k) = \int_k^\infty e^{-rT} (e^{-k} - e^k) q_T(s) ds \) is the value of a \( T \) maturity call option with strike price \( e^k \) \( (k=\ln K) \), and \( q_T(s) \) be the risk-neutral density of the log asset price \( s_T = \ln S_T \). As \( C_T(k) \) is not square integrable over \((-\infty, \infty)\), the introduction of a damping factor \( e^{\alpha k} \) aims at removing this problem.

Theorems 3.2 The Fourier transform of \( c_T(k) \) exist:

\[ \psi_T(\xi) = \int_{-\infty}^\infty e^{i\xi k} c_T(k) dk \]

Proof

\[ \psi_T(\xi) = \int_{-\infty}^\infty e^{i\xi k} \int_k^\infty e^{-rT} (e^{-k} - e^k) q_T(s) ds dk \]

\[ = \int_{-\infty}^\infty e^{-rT} f(l, v, \lambda, \tau; x = \xi - (\alpha + 1)i) \]

\[ \frac{e^{-rT} f(l, v, \lambda, \tau; x = \xi - (\alpha + 1)i)}{\alpha^2 + \alpha - \xi^2 + i(2\alpha + 1)\xi}, \quad \text{(11)} \]

where \( f \) is the characteristic function defined in theorem 3.1

A sufficient condition for \( c_T \) to be square-integrable is given by \( \psi_T(0) \) being finite. This is equivalent to \( E(S_T^{\alpha+1}) < \infty \).
Call prices can then be numerically obtained by using the inverse transform:

\[
C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ik\xi} \psi_T(\xi) \, d\xi
\]

\[
= \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-i\xi k} \psi_T(\xi) \, d\xi
\]

(12)

More precisely, the call price is determined by substituting (11) into (12) and performing the required integration. Integration (12) is a direct Fourier transform and lends itself to an application of the FFT.

References


