HYBRID DHAGE'S FIXED POINT THEOREM FOR ABSTRACT MEASURE INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

In this paper, an existence theorem for a abstract measure integro- differential equation is proved via a nonlinear alternative of Leray-Schauder type. An existence result is also proved for extremal solutions for Caratheodory as well as discontinuous cases of the nonlinearities involved in the equations.

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1. INTRODUCTION

The study of abstract measure differential equations is initiated by Sharma [14,15] and subsequently developed by Joshi [14], Shendge and Joshi [16] and Dhage [5,6] . Similarly, the study of abstract measure Integro- differential equation is studied by Dhadg[4], Dhage and Bellale [8] for various aspect of solutions. In such models of differential equations, the ordinary derivative is replaced by the derivative of the set functions which thereby gives the generalizations of the ordinary and measure differential equations. The various aspects of the solution of the abstract measure differential equations have been studied in the literature using the fixed point techniques such as Schauder's fixed point principle and Banach contraction mapping principle etc. In such situation one needs to show that the operator under consideration maps a certain set into itself. This is a serve restriction, which motivated us to persue the study of abstract measure Integro-differential equations using the nonlinear alternative of Leray – Schauder . In the present chapter we shall prove the existence and uniqueness result for an abstract measure integro- differential equation under Caratheodory condition. The existence of extremal solutions of the abstract measure-integro differential equation in question is also proved under certain monotonicity conditions of the nonlinearity involved in the equation. In the following section we give some preliminaries needed in the sequel.

2. DEFINITIONS AND NOTATIONS

Let X be a real Banach space with a norm $\|\cdot\|$. Let $x, y \in X$. Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{ z \in X \mid z = x + r(y - x),\ 0 \le r \le 1 \}. \tag{2.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$., we define the sets S_x and $\overline{S}_x$ in X by

$$S_x = \{ rx : -\infty < r < 1 \} \tag{2.2}$$

and

$$\overline{S}_x = \{ rx : -\infty < r \le 1 \}. \tag{2.3}$$

Thus we have

$$\overline{xy} = \overline{S}_y - S_x \text{ for all } x, y \in X.$$

Let $x_1, x_2 \in \overline{x_0 z}$ be arbitrary. We *say* $x_1 < x_2$ if $S_{x1} \subset \overline{S}_{x2}$ or equivalently $\overline{x_0\, x_1} \subset \overline{x_0\, x_2}$. In this case we also *write* $x_2 > x_1$.

Let M enote the σ-algebra of all subsets of X so that $ca(X, M)$ becomes a measurable space. Let $ca(X, M)$ be the space of all vector measures (signed measures) and define a norm $|\cdot|$ *on* $ca(X, M)$ by

$$\|p\| = |p|(X) \tag{2.4}$$

where $|p|$ is a total variation measure of p and is given by

$$|p|(X) = \sup_{\sigma} \sum_{i=1}^{\infty} |p(E_i)|, \quad \forall E_i \subset X. \tag{2.5}$$

It is known that $Ca(X, M)$ is a Banach space with respect to the norm $\|\cdot\|$ defined by (2.4). Let μ be a σ-finite measure on X and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $Bp(E) = 0$ for some $E \in M$. In this case we write $p << \mu$

For a fixed $x_0 \in X$, let M_0 denote the σ-algebra on S_{x0}. *Let* $z \in X$ be such that $z > x_0$ and *let* M_z denote the σ-algebra of all sets containing M_0 and the sets of the form S_x *for* $x \in \overline{x_0 z}$.. Finaly let $L^1_\mu\ (S_z, \mathbb{R})$ denote the space of all μ integrable rel valued function h on S_z with the norm $\|\ .\ \|_{L^1_\mu}$ defined by

$$\|\ .\ \|_{L^1_\mu} = \int_{S_x} |\, h(x)| d\mu\,.$$

3. STATEMENT OF THE PROBLEM

Let μ be a real or σ - finite positive measure on *X*. Given a $p \in ca(X, M)$ with $p << \mu$, consider the abstract measure differential equation (in short AMIDE)

$$\left.\begin{aligned} &\frac{dp}{d\mu} = f\Big(x, p(S_x) \int_{S_x} k\big(t,\ p(\bar{S}_t)\big)\, d\mu\,\Big) \quad \text{a.e. } [\,\mu\,] \text{ on } \overline{x_0 z}. \\ &p(E) = q(E), \ E \in M_0 \end{aligned}\right\} \tag{3.1}$$

where q is a given known vector measure, $\frac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ and $f : S_z \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is such that

$x \rightarrow f\big(x, p(S_x) \int_{S_x} k\big(t,\ p(\bar{S}_t)\big)\, d\mu\big)$ is μ-integrable for each $p \in ca(S_z, M_z)$.

Definition 3.1 *: Given an initial real measure q on M_0, a vector $p \in ca(S_z, M_z)$ ($z > x$) is said to be a solution of AMIDE (3.1) if*

(i) $p(E) = q(E)$, $E \in M_0$

(ii) (ii) $p << \mu$ on $\overline{x_0 z}$.,

(iii) p satisfies (2.3.1) a.e. [μ] on $\overline{x_0 z}$.

<u>**Remark**</u> **3.1 :** The AMIDE (3.1) is equivalent to the abstract measure integral equation

$$p(E) = \begin{cases} \displaystyle\int f\left(x, p(\bar{S}_x), \int_{S_x} k(t, p(\bar{S}_t)) d\mu\right) d\mu\,, \ E \in M_z, E \subset \overline{x_0 z}. \\[2ex] q(E), \quad E \in M_0\,. \end{cases} \tag{3.2}$$

A solution p of AMIDE (3.1) on $\overline{x_0 z}$.will be denoted by $p(S_{x0}, q)$.

In the following section we shall prove the main existence theorem for AMIDE (3.1) under suitable conditions on f. We shall use the following form of the Leray-Schauder's nonlinear alternative. See Dugundji and Granas [12].

Theorem 3.1 : Let $B(0, r)$ and $B[0, r]$ denote respectively the open and closed balls in a Banach space X centered at the origin 0 of radius r, for some $r > 0$. Let $T : B[0, r] \rightarrow X$ be a completely continuous operator. Then either

(i) the operator equation $Tx = x$ has a solution in $B[0, r]$, or

(ii) there exists an $u \in X$ *with* $\|u\| = r$ such that $u = \lambda T u$ for some $0 < \lambda < 1$.

4. EXISTENCE AND UNIQUENESS THEOREMS

We need the following definition in the sequel.

Definition 4.1 : *A function* $\psi : S_z \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ *is called a **Ɗ –function** if it satisfies*

(i) ψ is continuous ,

(ii) ψ is non decreasing , and

(iii) ψ is scalar submultiplicative , that is, $\psi(\lambda r) \leq \lambda(\psi r)$ for all $\lambda >$o and $r \in \mathbb{R}^+$

The class of all Ɗ- functions on $\mathbb{R}^+$ is denoted by Ψ .There do not exists Ɗ- functions on $\mathbb{R}$. Indeed the function $\psi : \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $\psi(r) = l\, r$, $l > 0$ satiesfies the conditions (i) – (iii) mentioned above and hence a Ɗ- functions on $\mathbb{R}^+$. Note that if $\psi \in \Psi$ then $\psi(0) = 0$

Definition 4.2: *A function* $f : S_z \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ *is said to satisfy condition of Caratheodory or simply Caratheodory if* *(i)* $x \to f(x, y_1, y_2)$ *is* μ*-measurable for each* $y \in \mathbb{R}$

(ii) $(y_1, y_2) \to f(x, y_1, y_2)$ *is continuous for almost everywhere* μ *on* $x \in \overline{x_0 z}$.,

Again , a Caratheodory function f is called L^1_μ *- Caratheodory.*

(iii) for each given real number $p > 0$ there exists a function $h_p \in L^1_\mu\ (S_z, \mathbb{R})$ such that $|f(x, y_1, y_2)| \le h_p(x)$ a.e. $[\mu]$ $x \in \overline{x_0 z}$.

for all $y_1, y_2 \in \mathbb{R}$ with $|y_1| \le p$ and $|y_2| \le p$

We consider the following set of assumptions.

(A_1) For any $z > x_0$, the σ-algebra M_z is compact with respect to the topology generated by the Pseudo-metric d defined by $d(E_1, E_2) = |\mu|(E_1 \Delta E_2)$, $E_1, E_2 \in M_z$.

(A_2) $\mu(\{x_0\}) = 0$.

(A_3) q is continuous on M_z with respect to the Pseudo-metric d defined in (A_1).

(A_4) The function $f(x, y_1, y_2)$ is L^1_μ - Caratheodory.

The function $k : J \times J \times \mathbb{R} \to \mathbb{R}$ is continuous function and there exists a function $\alpha \in L^1_\mu\ (J, \mathbb{R}^+)$ such that

$|k(x, y_1, y_2)| \le \alpha(x)\ |y|$ a.e. t, s. $\in J$ and $\forall\ y \in \mathbb{R}$.

(A_5) There exists a function $\phi \in L^1_\mu\ (S_z, \mathbb{R}^+)$ such that $\phi(x) > 0$

a.e. $[\mu]$, $x \in S_z$ and a continuous and nondecreasing function $\psi : [0, \infty) \to (0, \infty)$, such that

$|f(x, y_1, y_2)| \le \phi(x)\ \psi(|y_1| + |y_2|)$ a.e. $[\mu]$ on $\overline{x_0 z}$. For all $y_1, y_2 \in \mathbb{R}$.

We frequently make use of the following estimate concerning the multivalued function G(t, x, y) in the sequal .If the hypotheses (A_2)-(A_2) hold , then for any $x \in (J, \mathbb{R}^+)$ with $\|p\| \le r$, one has

Theorem 4.1 : *suppose that assumptions* (A_1)- (A_5) *hold. Further if there exists a real number* $r > 0$ *such that*

$$r > \|q\| + \|\phi\|_{L^1_\mu} \left(\left[1 + \|\alpha\|_{L^1_\mu}\right]\right) \psi(r) \tag{4.2}$$

then AMIDE (2.3.1) *has a solution on* M_z.

Proof : Let $X = ca(S_z, M_z)$ and consider an open ball $B(0, r)$ in $ca(S_z, M_z)$ centered at the origin and of radius r, where the real number $r > 0$ satisfies (4.2). Define an operator T from $B[0, r]$ into $ca(S_z, M_z)$ by

$$Tp(E) = \begin{cases} \int_E f\left(x, p(\bar{S}_x), \int_{S_x} k(t, p(\bar{S}_t))\right) d\mu, \\ E \in M_z,\ E \subset \overline{x_0 z} \\ q(E) \quad \text{if} \quad E \in M_0. \end{cases}$$

We shall show that the operator T satisfies all the conditions of Theorem 3.1 on $B[0, r]$.

Step I : First we show that T is a continuous on $B[0, r]$. Let $\{p_n\}$ be a sequence of vector measures in $B[0, r]$ converging to a vector measure p. Then by dominated convergence theorem,

$$\lim_n Tp_n(E) = \lim_{x \to \infty} \int_E f\left(x, p_n(\bar{S}_x), \int_{S_x} k(t, p_n(\bar{S}_t))\, d\mu\right) d\mu,$$

$$= \int_E f\left(x, p(\bar{S}_x), \int_{S_x} k(t, p(\bar{S}_t))\, d\mu\right) d\mu,$$

$$= Tp(E)$$

for all $E \in M_z$, $E \subset \overline{x_0 z}$. Similarly if $E \in M_0$, then

$$\begin{aligned} \lim Tp_n(E) &= q(E) \\ &= Tp(E) \end{aligned}$$

and, so T is a continuous operator on $B[0, r]$.

Step II : Next we show that $T(B[0, r])$ is a uniformly bounded and equi-continuous set in $ca(S_z, M_z)$. Let $p \in B[0, r]$ be arbitrary. Then we have $\| p \| \leq r$. Now by definition of the map T one has

$$Tp(E) = \begin{cases} \int_E f\left(x, p(\bar{S}_x), \int_{S_x} k(t, p(\bar{S}_t))\, d\mu\right) d\mu\,, \\ \\ E \in M_z,\ E \subset \overline{x_0 z},\ E \in M_0. \end{cases}$$

Therefore for any $E = F \cup G$, $F \in M_0$ and $G \in M_z$, $G \subset \overline{x_0 z}$,

we have, $| Tp(E) | \leq | q(E) | + \left| \int_E f\left(x, p(\bar{S}_x), \int_{S_x} k(t, p(\bar{S}_t))\, d\mu\right) d\mu\,, \right|$

$$\leq \| q \| + \int_E \left| f\left(x, p(\bar{S}_x), \int_{S_x} k(t, p(\bar{S}_t))\, d\mu\right) \right| d\mu\,,$$

$$\leq \| q \| + \| \phi \|_{L^1_\mu} \left(\left[1 + \| \alpha \|_{L^1_\mu} \right] \right) \psi(r)$$

for all $E \in M_z$. By definition of the norm $\| \cdot \|$ we have

$$\| Tp \| = | Tp | (S_z) \leq \| q \| + \| \phi \|_{L^1_\mu} \left(\left[1 + \| \alpha \|_{L^1_\mu} \right] \right) \psi(r)$$

This shows that the set $T(B[0, r])$ is uniformly bounded in $ca(S_z, M_z)$.

Now we show that $T(B[0, r])$ is an equi-continuous set in $ca(S_z, M_z)$. Let $E_1, E_2 \in M_z$. Then there are sets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_z$ with $G_1, G_2 \subset \overline{x_0 z}$., and $F_i \cap G_i = \phi$, for $i = 1, 2$.

We know the set-identities

$$\left.\begin{aligned} & G_1 = (G_1 - G_2) \cup (G_2 \cap G_1) \\ \textit{and}\quad & G_2 = (G_2 - G_1) \cup (G_2 \cap G_1). \end{aligned}\right\} \tag{4.3}$$

Therefore we have

$$\begin{aligned} Tp(E_1) - Tp(E_2) &= q(F_1) - q(F_2) \\ &\quad + \int_{G1-G2} f\left(x, p(\bar{S}_x), \int_{S_x} k(t, p(\bar{S}_t))\right) d\mu\,, \\ &\quad - \int_{G2-G1} f\left(x, p(\bar{S}_x), \int_{S_x} k(t, p(\bar{S}_t))\right) d\mu\,, \end{aligned}$$

Since $f(x, y)$ is L^1_μ - Caratheodory, we have that

$$\begin{aligned} | Tp(E_1) - Tp(E_2) | &\leq | q(F_1) - q(F_2) | \\ &\quad + \int_{G1 \Delta G2} \left| f\left(x, p(\bar{S}_x), \int_{S_x} k(t, p(\bar{S}_t))\, d\mu\right) \right| d\mu\,, \\ &\leq | q(F_1) - q(F_2) | + \int h_r(x)\, d\mu \end{aligned}$$

Assume that $d(E_1, E_2) = | \mu | (E_1 \Delta E_2) \to 0$.

Then we have $E_1 \to E_2$ and consequently $F_1 \to F_2$ and $| \mu | (G_1 \Delta G_2) \to 0$. From the continuity of q on M_0 it follows that

$$\begin{aligned} | Tp(E_1) - Tp(E_2) | &\leq | q(F_1) - q(F_2) | + \int h_r(x)\, d\mu \\ &\to 0 \text{ as } E_1 \to E_2. \end{aligned}$$

This shows that $T(B[0, r])$ is an equi-continuous set in $ca(S_z, M_z)$. Now $T(B[0, r])$ is uniformly bounded and equi-continuous set in $ca(S_z, M_z)$, so it is compact in the norm topology on $ca(S_z, M_z)$. Now an application of Arzela – Ascolli theorem yields that $T(B[0, r])$ is a compact subset of $ca(S_z, M_z)$. As a result T is a continuous and totally bounded operator on $B[0, r]$. Hence an application of Theorem 3.1 yields that either $x = Tx$ has a solution or the operator equation $x = \lambda Tx$ has a solution u with $\| u \| = r$ for some $0 < \lambda < q$. We shall show that this laster asserstion is not possible. We assume the contrary. Then there is an $u \in X$ with $\| u \| = r$ satisfying $u = \lambda Tu$ for some λ, $0 < \lambda < 1$. Now for any $E \in M_z$, we have $E = F \cup G$, where $F \in M_0$ and $G \subset \overline{x_0 z}$, satisfying $F \cap G = \phi$.

Now, $u(E) = \lambda T u(E)$

$$u(E) = \begin{cases} \lambda q(F), & F \in M_0 \\ \lambda \int_G f\left(x, u(\bar{S}_x), \int_{\bar{S}_x} k(t, u(\bar{S}_t))\, d\mu\right) d\mu, & G \in M_z \text{ and } G \subset \overline{x_0 z}, \end{cases}$$

$$\begin{aligned} \text{Therefore, } | u(E) | &= | \lambda q(F) | + \left| \lambda \int_G f\left(x, u(\bar{S}_x), \int_{\bar{S}_x} k(t, u(\bar{S}_t))\, d\mu\right) d\mu \right| \\ &\le \| q \| + \left| \int_G f\left(x, u(\bar{S}_x), \int_{\bar{S}_x} k(t, u(\bar{S}_t))\, d\mu\right) d\mu \right| \\ &\le \| q \| + \int_G \phi(t)\, \psi(| u(\bar{S}_t) |)\, d\mu \\ &\le \| q \| + \int_G \phi(t)\, \psi(\| u \|)\, d\mu \\ &= \| q \| + \| \phi \|_{L^1_\mu}\, \psi(\| u \|). \end{aligned}$$

This further implies that

$$\begin{aligned} \| \mu \| &= | u | (S_z) \le | u(E) | \\ &\le \| q \| + \| \phi \|_{L^1_\mu}\, \psi(\| u \|). \end{aligned}$$

Substituting $\| u \| = r$ in the above inequality yields that

$$r \le \| q \| + \| \phi \|_{L^1_\mu}\, \psi(r). \tag{4.4}$$

which is a contradiction to the inequality (4.2).

Hence the operator equation $p = Tp$ has a solution in v with $\| v \| \le r$. Consequently the AMIDE (3.1) has a solution $p = p(S_{x0}, q)$ in $B[0, r]$. This completes the proof.

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