ON SMALL PRINCIPALLY INJECTIVE RINGS



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ON SMALL PRINCIPALLY INJECTIVE RINGS



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ABSTRACT

The purposes of this thesis are (1) to study properties and characterizations of small principally injective modules, (2) to study properties and characterizations of small principally injective rings, and (3) to find some relations between small principally injective modules, small principally injective rings and projective modules.

Let *R* be a ring. A right *R*-module *M* is called *principally injective* if every *R*-homomorphism from a principal right ideal of *R* to *M* can be extended to an *R*-homomorphism from *R* to *M*. A right *R*-module *M* is called *small principally injective* if every *R*-homomorphism from a small and principal right ideal of *R* to *M* can be extended to an *R*-homomorphism from *R* to *M*. *R* is called a right *small principally injective if injective ring* if R_R is a small principally injective module.

The results were as follows. (1) Let *R* be a right small principally injective ring. Then (1.1) lr(Ra) = Ra for any $a \in J(R)$. (1.2) If $aR \oplus bR$ and $Ra \oplus Rb$ are both direct, $a, b \in J(R)$, then l(a)+l(b) = R. (2) Let *R* be right small principally injective, $a \in R$ and $b \in J(R)$. (2.1) If bR embeds in aR, then Rb is an image of Ra. (2.2) If aR is an image of bR, then Ra embeds in Rb. (2.3) If $bR \cong aR$, then $Ra \cong Rb$. (3) The following conditions are equivalent for a ring R : (3.1) every small and principally injective module is small principally injective; (3.2) every factor module of a small principally injective and $b_i \in J(R)$, $(1 \le i \le n)$. (4.1) If $Rb_1 \oplus ... \oplus Rb_n$, is direct, then any $\alpha : Rb_1 + ... + Rb_n \to R$ can be extended to R . (4.2) If $b_1R \oplus ... \oplus b_nR$ is direct, then $R(b_1 + ... + b_n) = Rb_1 + ... + Rb_n$.

Keywords: principally injective rings, small principally injective rings

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Table of Contents

Abstract	(3)
Acknowledgements	(4)
Table of Contents	(5)
List of Abbreviations	(7)
CHAPTER	
1 INTRODUCTION	
1.1 Background and Statement of the Problems	9
1.2 Purpose of the Study	9
1.3 Research Questions and Hypothesis	10
1.4 Theoretical Perspective	10
1.5 Delimitations and Limitations of the Study	10
1.6 Significance of the Study	10
2 LITERATURE REVIEW	
2.1 Rings, Modules, Submodules and Endomorphism Rings	11
2.2 Essential and Superfluous Submodules	15
2.3 Annihilators and Singular Modules	16
2.4 Maximal and Minimal Submodules	17
2.5 Injective and Projective Modules	18
2.6 Direct Summands and Product of Modules	19
2.7 Generated and Cogenerated Classes	22
2.8 The Trace and Reject.	23
2.9 Socle and Radical of Modules	24
2.10 The Radical of a Ring	25
3 RESEARCH RESULT	
3.1 SP-injective Modules	27
3.2 SP-injective Rings	31

Table of Contents (Continued)

List of References	
Appendix	
Appendix A	
Biography	



List of Abbreviations

$A \oplus B$	A direct sum B
$End_R(M)$	The set of R -homomorphism from M to M
F	Field F
$f: M \to N$	A function f from M to N
f(M), $Im(f)$	Image of f
$Hom_R(M,N)$	The set of R -homomorphism from M to N
Ker(f)	Kernel of f
$J(M)$, $Rad(M_R)$	Jacobson radical of a right R -module M
$J(R) = Rad(R_R)$	Jacobson radical of a ring R
J(S)	Jacobson radical of a ring S
$J(S) \subset {}_SS_S$	J(S) is an (two-side) ideal of ring S
$l_M(A)$	Left annihilator of A in M
M_R	M is a right R-module
$M_1 \times M_2$	Cartesian products of M_1 and M_2
M/K	A factor module of M modulo K or a factor module of M by K
$M \cong N$	M isomorphic N
R	Ring R
R_R	Ring <i>R</i> is a right <i>R</i> -module is called Regular right <i>R</i> -module
$r_R(X)$	Right annihilator of X in R
Z(M)	Singular submodule of M
1_M	Identity map on a module M
$\begin{pmatrix} F & F \\ F & F \end{pmatrix} = M_2(F)$	The set of all 2×2 matrices having elements of a field <i>F</i> as entries
$\eta: M \to M/K$	η (<i>eta</i>) is the natural epimorphism of <i>M</i> onto <i>M</i> / <i>K</i>
$\iota = \iota_{A_{\subset}B} : A \longrightarrow B$	i (<i>iota</i>) is the inclusion map of A in B
π_{j}	π_j is the <i>j</i> -th projection map
\forall	For all
\cap	Intersection of set

List of Abbreviations (Continued)



CHAPTER 1 INTRODUCTION

In modules and rings theory research field, there are three methods for doing the research. Firstly, to study about the fundamental of algebra and modules theory over arbitrary rings. Secondly, to study about the modules over special rings. Thirdly, to study about ring R by way of the categories of R-modules. Many mathematicians have concentrated on these methods.

1.1 Background and Statement of the Problems

Many generalizations of the injectivity were obtained, e.g., *principally injectivity* and *mininjectivity*. In [2], V. Camillo introduced the definition of principally injective modules by calling a right *R*-module *M* is *principally injective* if every *R*homomorphism from a principal right ideal of *R* to *M* can be extended to an *R*homomorphism from *R* to *M*. In [7], Nicholson and Yousif studied to the structure of principally injective rings and gave some applications of these rings. A ring *R* is called *right principally injective* if every *R*-homomorphism from a principal right ideal of *R* to *R* can be extended to an *R*-homomorphism from *R* to *R*. In [12], L.V. Thuyet, and T.C. Quynh introduced the definitions of a small principally module. A right *R*-module *M* is called *small principally injective* if every *R*-homomorphism from a small and principal right ideal *aR* to *M* can be extended to an *R*-homomorphism from *R* to *M*. In [10], N. V. Sanh, K. P. Shum, S. Dhompongsa and S. Wongwai introduced the definitions of quasi principally injective modules. A right *R*-module *M* is called *quasi-principally injective* if every *R*-homomorphism from an *M*-cyclic submodule of *M* to *M* can be extended to *M*.

1.2 Purpose of the Study

In this thesis, we have the purposes of study which are to extend concept of the previous works and to generalize new concepts which are :

1.2.1 To extend the concept of *principally injective rings*.

1.2.2 To generalize the concept of *small principally injective modules*.

1.2.3 To establish and extend some new concepts which are dual to *small principally-injective rings* and *small principally-injective modules*.

1.3 Research Questions and Hypothesis

We are interested in seeing to extend the characterizations and properties which remain valid from these previous concepts which can be extended from *principally injective rings, principally quasi-injective modules* [9], and *small -injective rings* [12]. In this research, we give characterizations and properties of these modules. A right *R*-module *M* is called *small principally injective* if every *R*-homomorphism from a small and principal right ideal *aR* to *M* can be extended to an *R*-homomorphism from *R* to *M*. If R_R is *SP*-injective modules, then we call *R* is *SP*-injective rings. In this research we give some properties and characterizations of *SP*-injective modules and *SP*injective rings.

1.4 Theoretical Perspective

In this thesis, we use many of the fundamental theories which are concerned to the rings and modules research. By the concerned theories are :

1.4.1 The fundamental of algebra theories.

1.4.2 The basic properties of rings and modules theory.

1.5 Delimitations and Limitations of the Study

For this thesis, we have the scopes and the limitations of studying which are concerned to the previous works which are:

1.5.1 To study properties and and characterizations of SP-injective modules.

1.5.2 To study properties and and characterizations of SP-injective rings.

1.6 Significance of the Study

The advantage of education and studying in this research, we can improve and develop the concepts and knowledge in the algebra and modules research field.

CHAPTER 2 LITERATURE REVIEW

In this chapter we give notations, definitions and fundamental theories of the modules and rings theory which are used in this thesis.

2.1 Rings, Modules, Submodules and Endomorphism Rings

This section is assembled summary of various notations, terminology and some background theories which are concerned and used for this thesis.

2.1.1 Definition. [14] By a *ring* we mean a nonempty set R with two binary operations + and •, called *addition* and *multiplication* (also called *product*), respectively, such that

(1) (R, +) is an additive abelian group.

(2) (R, \bullet) is a multiplicative semigroup.

(3) Multiplication is distributive (on both sides) over addition; that is, for all $a, b, c \in R$, $a^{\bullet}(b + c) = a^{\bullet}b + a^{\bullet}c$ and $(a + b)^{\bullet}c = a^{\bullet}c + b^{\bullet}c$.

The two distributive laws are respectively called the *left distributive* law and the *right distributive* law.

A commutative ring is a ring R in which multiplication is commutative; i.e. if $a \cdot b = b \cdot a$ for all $a, b \in R$. If a ring is not commutative it is called noncommutative.

A ring with unity is a ring R in which the multiplicative semigroup (R, \bullet) has an identity element; that is, there exists $e \in R$ such that ea = a = ae for all $a \in R$. The element e is called *unity* or the *identity* element of R. Generally, the unity or identity element is denoted by 1. In this thesis, R will be an associative ring with identity.

> **2.1.2 Definition.** [14] A nonempty subset *I* of a ring *R* is called an *ideal* of *R* if (1) $a, b \in I$ implies $a - b \in I$.

(2) $a \in I$ and $r \in R$ imply $ar \in I$ and $ra \in I$.

2.1.3 Definition. [13] A subgroup *I* of (R, +) is called a *left ideal* of *R* if $RI \subset I$, and a *right ideal* if $IR \subset I$.

2.1.4 Definition. [14] A right ideal *I* of a ring *R* is called *principal* if I = aR for some $a \in R$.

2.1.5 Definition. [14] Let R be a ring, M an additive abelian group and $(m, r) \mapsto mr$, a mapping of $M \times R$ into M such that

(1)
$$mr \in M$$

(2) $(m_1 + m_2)r = m_1r + m_2r$
(3) $m(r_1 + r_2) = mr_1 + mr_2$
(4) $(mr_1)r_2 = m(r_1r_2)$
(5) $m \cdot 1 = m$

for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. Then M is called a *right R-module*, often written as M_R . Often mr is called the *scalar multiplication* or just *multiplication* of m by r on right. We define left R-module similarly.

2.1.6 Definition. [13] Let *M* be a right *R*-module. A subgroup *N* of (M, +) is called a *submodule* of *M* if *N* is closed under multiplication with elements in *R*, that is $nr \in N$ for all $n \in N$, $r \in R$. Then *N* is also a right *R*-module by the operations induced from $M : N \times R \to N$, $(n, r) \mapsto nr$, for all $n \in N$, $r \in R$.

2.1.7 Proposition. A subset N of an R-module M is a submodule of M if and

only if
(1)
$$0 \in N$$
.
(2) $n_1, n_2 \in N$ implies $n_1 - n_2 \in N$.
(3) $n \in N, r \in R$ implies $nr \in N$.

Proof. See [15, Lemma 5.3].

2.1.8 Definition. [1] Let *M* be a right *R*-module and let *K* be a submodule of *M*. Then the set of cosets

$$M/K = \{ x + K \mid x \in M \}$$

is a right *R*-module relative to the addition and scalar multiplication defined via

(x+K) + (y+K) = (x+y) + K and (x+K)r = xr + K.

The additive identity and inverses are given by

K = 0 + K and -(x + K) = -x + K.

The module M/K is called (the *right R-factor module of*) M modulo K or the factor module of M by K.

2.1.9 Definition. [13] Let *M* and *N* be right *R*-modules. A function $f : M \to N$ is called an (*R*-module) homomorphism if for all $m, m_1, m_2 \in M$ and $r \in R$

$$f(m_1r + m_2) = f(m_1)r + f(m_2).$$

Equivalently, $f(m_1 + m_2) = f(m_1) + f(m_2)$ and f(mr) = f(m)r.

The set of *R*-homomorphisms of *M* in *N* is denoted by $Hom_R(M, N)$. In particular, with this addition and the composition of mappings, $Hom_R(M, M) = End_R(M)$ becomes a ring, called the *endomorphism ring* of *M* and $f \in End_R(M)$ is called *an R-endomorphism*. [13, 6.4]

2.1.10 Definition. [1] Let $f: M \to N$ be an *R*-homomorphism. Then

- (1) f is called *R*-monomorphism (or *R*-monic) if f is injective (one-to-one).
- (2) f is called *R*-epimorphism (or *R*-epic) if f is surjective (onto).

(3) f is called *R*-isomorphism if f is bijective (one-to-one and onto).

Two modules *M* and *N* are said to be *R*-isomorphic, abbreviated $M \cong N$ in case there is an *R*-isomorphism $f: M \to N$.

2.1.11 Definition. [1] Let *K* be a submodule of *M*. Then the mapping $\eta_K : M \to M/K$ from *M* onto the factor module *M/K* defined by

$$\eta_K(x) = x + K \in M/K \qquad (x \in M)$$

is seen to be an *R*-epimorphism with kernel *K*. We call η_K the *natural epimorphism of M* onto *M*/*K*.

2.1.12 Definition. [1] Let $A \subset B$. Then the function $\iota = \iota_{A \subset B} : A \to B$ defined by $\iota = (1_{B|A}) : a \mapsto a$ for all $a \in A$ is called the *inclusion map* of A in B. Note that if $A \subset B$ and $A \subset C$, and if $B \neq C$, then $\iota_{A \subset B} \neq \iota_{A \subset C}$. Of course $1_A = \iota_{A \subset A}$.

2.1.13 Definition. [14] Let *M* and *N* be right *R*-modules and let $f : M \to N$ be an *R*-homomorphism. Then the set

$$Ker(f) = \{ x \in M | f(x) = 0 \}$$
 is called the *kernel* of f

and

 $f(M) = \{ f(x) \in N \mid x \in M \}$ is called the *homomorphic image* (or simply *image*) of *M* under *f* and is denoted by Im(f).

2.1.14 Proposition. Let M and N be right R-modules and let $f : M \to N$ be an R-homomorphism. Then

- (1) Ker(f) is a submodule of M.
- (2) Im(f) = f(M) is a submodule of N.

Proof. See [13, 6.5].

2.1.15 Proposition. Let M and N be right R-modules and let $f: M \to N$ be an *R*-isomorphism. Then the inverse mapping $f^{-1}: N \to M$ is an *R*-isomorphism.

Proof. See[14, Chapter14, 3].

2.1.16 Theorem. Let M, M', N and N' be right R-modules and let $f : M \to N$ be an R-homomorphism.

(1) If $g: M \to M'$ is an epimorphism with $Ker(g) \subset Ker(f)$, then there exists a unique homomorphism $h: M' \to N$ such that

f = hg.

Moreover, Ker(h) = g(Ker(f)) and Im(h) = Im(f), so that h is monic if and only if Ker(g) = Ker(f) and h is epic if and only if f is epic.

(2) If $g : N' \to N$ is a monomorphism with $Im(f) \subset Im(g)$, then there exists a unique homomorphism $h : M \to N'$ such that

f = gh.

Moreover, Ker(h) = Ker(f) and $Im(h) = g^{\leftarrow}(Im(f))$, so that h is monic if and only if f is monic and h is epic if and only if Im(g) = Im(f).



2.1.17 Definition. [20] A submodule *K* of the module *M* is fully invariant in *M* if $f(K) \subset K$ for every endomorphism *f* of *M*.

2.2 Essential and Superfluous Submodules

In this section, we give the definitions of essential and superfluous submodules and some theories which are used in this thesis.

2.2.1 Definition. [13] A submodule *K* of *M* is called *essential* (or *large*) in *M*, abbreviated $K \subset^e M$, if for every submodule *L* of *M*, $K \cap L = 0$ implies L = 0.

2.2.2 Definition. [13] A submodule K of M is called *superfluous* (or *small*) in M, abbreviated $K \ll M$, if for every submodule L of M, K + L = M implies L = M.

2.2.3 Proposition. Let M be a right R-module with submodules $K \subset N \subset M$ and $H \subset M$. Then

(1) $N \ll M$ if and only if $K \ll M$ and $N/K \ll M/K$;

(2) $H + K \ll M$ if and only if $H \ll M$ and $K \ll M$.

Proof. See [1, Proposition 5.17].

2.2.4 Proposition. If $K \ll M$ and $f: M \to N$ is a homomorphism then $f(K) \ll N$. In particular, if $K \ll M \subset N$ then $K \ll N$.

Proof. See [1, Proposition 5.18].

2.3 Annihilators and Singular Modules

In this section, we give the definitions of annihilators, singular modules and some theories which are used in this thesis.

2.3.1 Definition. [1] Let *M* be a right (resp. left) *R*-module. For each $X \subset M$, the *right* (resp. *left*) *annihilator* of *X* in *R* is defined by

 $r_R(X) = \{ r \in R \mid xr = 0, \forall x \in X \} (\text{resp. } l_R(X) = \{ r \in R \mid rx = 0, \forall x \in X \}).$ For a singleton $\{x\}$, we usually abbreviated to $r_R(x)$ (resp. $l_R(x)$).

2.3.2 Proposition. Let M be a right R-module, let X and Y be subsets of M and let A and B be subsets of R. Then

- (1) $r_R(X)$ is a right ideal of R.
- (2) $X \subset Y$ imples $r_R(Y) \subset r_R(X)$.
- (3) $A \subset B$ imples $l_M(B) \subset l_M(A)$.
- (4) $X \subset l_M r_R(X)$ and $A \subset r_R l_M(A)$.

Proof. See [1, Proposition 2.14 and Proposition 2.15].

2.3.3 Proposition. Let M and N be right R-modules and let $f : M \to N$ be a homomorphism. If N' is an essential submodule of N, then $f^{-1}(N')$ is an essential submodule of M.

Proof. See [4, Lemma 5.8(a)].

2.3.4 Proposition. Let M be a right R-module over an arbitrary ring R, the set $Z(M) = \{ x \in M \mid r_R(x) \text{ is essential in } R_R \}$ is a submodule of M.

Proof. See [4, Lemma 5.9].

 \square

2.3.5 Definition. [4] The submodule $Z(M) = \{ x \in M \mid r_R(x) \text{ is essential in } R_R \}$ is called the *singular submodule* of *M*. The module *M* is called a *singular module* if Z(M) = M. The module *M* is called *a nonsingular module* if Z(M) = 0.

2.4 Maximal and Minimal Submodules

In this section, we give the definitions and some properties of maximal submodules, minimal (or simple) submodules and some theories which are used in this thesis.

2.4.1 Definition. [13] A right *R*-module *M* is called *simple* if $M \neq 0$ and *M* has no submodules except 0 and *M*.

2.4.2 Definition. [13] A submodule K of M is called *maximal submodule* of M if $K \neq M$ and it is not properly contained in any proper submodules of M, i.e. K is *maximal in M* if, $K \neq M$ and for every $A \subset M$, $K \subset A$ implies K = A.

2.4.3 Definition. [13] A submodule N of M is called *minimal* (or *simple*) submodule of M if $N \neq 0$ and it has no non zero proper submodules of M, i.e. N is *minimal* (or *simple*) in M if $N \neq 0$ and for every nonzero submodules A of M, $A \subset N$ implies A = N.

2.4.4 Proposition. Let M and N be right R-modules. If $f : M \to N$ is an epimorphism with Ker(f) = K, then there is a unique isomorphism $\sigma : M/K \to N$ such that $\sigma(m+K) = f(m)$ for all $m \in M$

Proof. See [1, Corollary 3.7].

2.4.5 Proposition. Let K be a submodule of M. A factor module M/K is simple if and only if K is a maximal submodule of M.

Proof. See [1, Corollary 2.10].

2.5 Injective and Projective Modules

In this section, we give the definitions of the injective modules, injective testing, projective modules and some theories which are used in this thesis.

2.5.1 Definition. [1] Let *M* be a right *R*-module. A right *R*-module *U* is called *injective relative to M* (or *U is M-injective*) if for every submodule *K* of *M*, for every homomorphism $\varphi : K \to U$ can be extended to a homomorphism $\alpha : M \to U$.

A right R-module U is said to be *injective* if it is M-injective for every right R-module M.

2.5.2 Proposition. The following statements about a right R-module U are equivalent :

(1) *U* is injective;

(2) U is injective relative to R;

(3) For every right ideal $I \subset R_R$ and every homomorphism $h : I \to U$ there exists an $x \in U$ such that h is left multiplicative by x

h(a) = xa for all $a \in I$.

Proof. See [1, 18.3, Baer's Criterion].

2.5.3 Definition. [1] Let *M* be a right *R*-module. A right *R*-module *U* is called *projective relative to M* (or *U is M-projective*) if for every N_R , every epimorphism $g: M_R \rightarrow N_R$, for every homomorphism $\gamma: U_R \rightarrow N_R$ can be lifted to an *R*-homomorphism $\hat{\gamma}: U \rightarrow M$. A right *R*-module *U* is said to be *projective* if it is projective for every right *R*-module *M*.

2.5.4 Proposition. Every right (resp. left) R-module can be embedded in an injective right (resp. left) R-module.

Proof. See [1, Proposition 18.6].

2.6 Direct Summands and Product of Modules

Given two modules M_1 and M_2 we can construct their Cartesian product $M_1 \times M_2$. The structure of this product module is then determined "co-ordinatewise" from the factors $M_1 \times M_2$. For this section we give the definitions of direct summand, the projection and the injection maps, product of modules and some theories which are used in this thesis.

2.6.1 Definition. [1] Let *M* be a right *R*-module. A submodule *X* of *M* is called a *direct summand* of *M* if there is a submodule *Y* of *M* such that $X \cap Y = 0$ and X + Y = M. We write $M = X \oplus Y$; such that *Y* is also a *direct summand*.

2.6.2 Definition. [1] Let M_1 and M_2 be *R*-modules. Then with their products module $M_1 \times M_2$ are associated the natural injections and projections

$$\varphi_j \colon M_j \to M_1 \times M_2$$
 and $\pi_j \colon M_1 \times M_2 \to M_j$

(j = 1, 2), are defined by

Moreover, we have

$$\varphi_1(x_1) = (x_1, 0), \qquad \varphi_2(x_2) = (0, x_2)$$

and

$$\pi_1(x_1, x_2) = x_1,$$
 $\pi_2(x_1, x_2) = x_2$
 $\pi_1 \varphi_1 = 1_{M_1}$ and $\pi_2 \varphi_2 = 1_{M_2}$

2.6.3 Definition. [1] Let A be a direct summand of M with complementary direct summand B, so $M = A \oplus B$. Then

 $\pi_A: a+b \mapsto a \ (\ a \in A, b \in B \)$

defines an epimorphism $\pi_A : M \to A$ is called *the projection of M on A along B*.

2.6.4 Definition. [13] Let $\{A_i, i \in I\}$ be a family of objects in the category C. An object *P* in C with morphisms $\{\pi_i : P \to A_i\}$ is called the *product* of the family $\{A_i, i \in I\}$ if :

For every family of morphisms $\{f_i : X \to A_i\}$ in the category C, there is

a unique morphism $f: X \to P$ with $\pi_i f = f_i$ for all $i \in I$.

For the object *P*, we usually write $\prod_{i \in I} A_i$, $\prod_I A_i$ or $\prod A_i$. If all A_i are equal to *A*, then we put $\prod_I A_i = A^I$.

The morphism π_i are called the *i-projections* of the product. The definition can be described by the following commutative diagram :



2.6.5 Definition. [13] Let $\{M_i, i \in I\}$ be a family of *R*-modules and $(\prod_{i \in I} M_i, K_i)$

 π_i) the product of the M_i . For $m, n \in \prod_{i \in I} M_i, r \in R$, using

 $\pi_i(m+n) = \pi_i(m) + \pi_i(n)$ and $\pi_i(mr) = \pi_i(m)r$,

a right *R*-module structure is defined on $\prod_{i \in I} M_i$ such that the π_i are homomorphisms.

With this structure $(\prod_{i \in I} M_i, \pi_i)$ is the product of the $\{M_i, i \in I\}$ in *R*-module.

2.6.6 Proposition. Properties:

(1) If $\{f_i : N \to M_i, i \in I\}$ is a family of morphisms, then we get the map $f : N \to \prod_{i \in I} M_i$ such that $n \mapsto (f_i(n))_{i \in I}$

and $Ker(f) = \bigcap_{I} Ker(f_i)$ since f(n) = 0 if and only if $f_i(n) = 0$ for all $i \in I$.

(2) For every $j \in I$, we have a canonical embedding

 $\varepsilon_j: M_j \to \prod_{i \in I} M_i$, such that $m_j \mapsto (m_j \delta_{ji})_{i \in I}, m_j \in M_j$,

with $\varepsilon_j \pi_j = 1_{M_j}$, i.e. π_j is a retraction and ε_j a coretraction.

This construction can be extended to larger subsets of I : For a subset A

 $\subset I$ we form the product $\prod_{i \in A} M_i$ and a family of homomorphisms

$$f_j: \prod_{i \in A} M_i \to M_j, \qquad f_j = \begin{cases} \pi_j \text{ for } j \in A, \\ 0 \text{ for } j \in I - A. \end{cases}$$

Then there is a unique homomorphism

$$\varepsilon_{A}: \prod_{i \in A} M_{i} \to \prod_{i \in I} M_{i} \text{ with } \varepsilon_{A} \pi_{j} = \begin{cases} \pi_{j} \text{ for } j \in A, \\ 0 \text{ for } j \in I - A. \end{cases}$$

The universal property of $\prod_{i \in A} M_i$ yields a homomorphism

$$\pi_{A}: \prod_{i \in I} M_{i} \to \prod_{i \in A} M_{i} \text{ with } \pi_{A} \pi_{j} = \pi_{j} \text{ for } j \in I.$$

Together this implies $\varepsilon_A \pi_A \pi_j = \varepsilon_A \pi_j = \pi_j$ for all $j \in I$, and by the properties of the product

 $\prod_{i \in A} M_i, we get \varepsilon_A \pi_A = 1_{M_A}.$

Proof. See [13, 9.3, Properties (1), (2)]

2.6.7 Definition. [1] We say $(M_{\alpha})_{\alpha \in A}$ is independent in case for each $\alpha \in A$

$$M_{\alpha} \cap (\sum_{\beta \neq \alpha} M_{\beta}) = 0$$

If the submodules $(M_{\alpha})_{\alpha \in A}$ of *M* are independent, we say that the sum $\sum_{A} M_{\alpha}$ is *direct* and write

$$\sum_{A} M_{\alpha} = \bigoplus_{A} M_{\alpha} \, .$$

2.6.8 Proposition. [1] Let $(M_{\alpha})_{\alpha \in A}$ be an indexed set of submodules of a module M with inclusion maps $(i_{\alpha})_{\alpha \in A}$. Then the following are equivalent:

- (a) ∑_A M_α is the internal direct sum of (M_α)_{α∈A};
 (b) i = ⊕_Ai_α : ⊕_A M_α → M is monic;
- (c) $(M_{\alpha})_{\alpha \in A}$ is independent;

(d) $(M_{\alpha})_{\alpha \in F}$ is independent for every finite subset $F \subset A$;

(e) For every pair $B, C \subset A$, if $B \cap C = \emptyset$, then $(\sum_{B} M_{\beta}) \cap (\sum_{C} M_{\gamma}) = 0$.

Proof. See [1, Proposition 6.10].

2.7 Generated and Cogenerated Classes

In this section, we give some definitions and theories of the generated and cogenerated classes which are concerned in this thesis.

2.7.1 Definition. [13] A subset X of a right *R*-module *M* is called a *generating* set of *M* if XR = M. We also say that X generates *M* or *M* is generated by X. If there is a finite generating set in *M*, then *M* is called *finitely generated*.

2.7.2 Definition. [1] Let U be a class of right *R*-modules. A module *M* is (*finitely*) generated by U (or U (*finitely*) generates M) if there exists an epimorphism

$$\bigoplus_{i \in I} U_i \to M$$

for some (finite) set *I* and $U_i \in U$ for every $i \in I$.

If $U = \{U\}$ is a singleton, then we say that *M* is (*finitely*) generated by U or (*finitely*) *U*-generates; this means that there exists an epimorphism

 $U^{(1)} \rightarrow M$

for some (finite) set I.

2.7.3 Proposition. If a module M has a generating set $L \subset M$, then there exists an epimorphism

$$R^{(L)} \rightarrow M$$

Moreover, M is finitely R-generated if and only if M is finitely generated.

Proof. See [1, Theorem 8.1].

. . .

2.7.4 Definition. [17] Let *M* be a right *R*-module. A submodule *N* of *M* is said to be an *M*-cyclic submodule of *M* if it is the image of an endomorphism of *M*.

2.7.5 Definition. [1] Let U be a class of right *R*-modules. A module *M* is (finitely) cogenerated by U (or U (finitely) cogenerates M) if there exists a monomorphism

$$M \to \prod_{i \in I} U_i$$

for some (finite) set *I* and $U_i \in U$ for every $i \in I$.

If $U = \{U\}$ is a singleton, then we say that a module M is (*finitely*) cogenerated by U or (finitely) U-cogenerates; this means that there exists a monomorphism

$$M \rightarrow U^{I}$$

for some (finite) set I.

2.8 The Trace and Reject

In this section, we give some definitions and theories of the trace and reject which are concerned in this thesis.

2.8.1 Definition. [1] Let U be a class of right R-modules. The trace of U in M and the *reject* of U in M are defined by

$$Tr_M(\mathbf{U}) = \sum \{ Im(h) \mid h : U \to M \text{ for some } U \in \mathbf{U} \}$$

and

$$Rej_M(\mathbf{U}) = \bigcap \{ Ker(h) \mid h : M \to U \text{ for some } U \in \mathbf{U} \}.$$

If $U = \{U\}$ is a singleton, then the trace of U in M and the reject of U in M are $Tr_M(U) = \sum \{ Im(h) \mid h \in Hom_R(U, M) \}$ in the form

and

$$Rej_M(U) = \bigcap \{ Ker(h) \mid h \in Hom_R(M, U) \}.$$

2.8.2 Proposition. Let U be a class of right R-modules and let M be a right Rmodule.Then

> (1) Tr_M (U) is the unique largest submodule L of M generated by U; (2) $Rej_M(U)$ is the unique smallest submodule K of M such that M/K is

cogenerated by U.

Proof. See [1, Proposition 8.12].

2.9 Socle and Radical of Modules

In this section, we give some definitions and theories of the socle and radical of modules which are used in this thesis.

2.9.1 Definition. [13] Let *M* be a right *R*-module. The *socle* of *M*, *Soc*(*M*), we denote the sum of all simple submodules of *M*. If there are no simple submodules in *M* we put Soc(M) = 0.

2.9.2 Definition. [13] Let *M* be a right *R*-module. The *radical* of *M*, Rad(M), we denote the intersection of all maximal submodules of *M*. If *M* has no maximal submodules we set Rad(M) = M.

2.9.3 Proposition. Let ε be the class of simple R-modules and let M be an R-module. Then

Soc(M) =
$$Tr_M(\varepsilon)$$

= $\cap \{ L \subset M \mid L \text{ is essential in } M \}.$

Proof. See [13, 21.1].

2.9.4 Proposition. Let ε be the class of simple *R*-modules and let *M* be an *R*-

module. Then

$$Rad(M) = Rej_M(\varepsilon)$$

= $\Sigma \{ L \subset M \mid L \text{ is superfluous in } M \}.$

Proof. See [13, 21.5].

2.9.5 Proposition. Let M be a right R-module. A right R-module M is finitely generated if and only if $Rad(M) \ll M$ and M/Rad(M) is finitely generated. **Proof.** See [13, 21.6, (4)].

2.9.6 Proposition. Let M be a right R-module. Then $Soc(M) \subset^{e} M$ if and only if every non-zero submodule of M contains a minimal submodule.

Proof. See [1, Corollary 9.10].

2.10 The Radical of a Ring

In this section, we give some definitions and theories of the radical of a ring which are used in this thesis.

2.10.1 Definition. [1] Let *R* be a ring. The radical $Rad(R_R)$ of R_R is an (two side) ideal of *R*. This ideal of *R* is called the (*Jacobson*) *radical* of *R*, and we usually abbreviated by

$$J(R) = Rad(R_R).$$

Since R = 1R is finite generated, $J(R) \ll R$. If $a \in J(R)$, then $aR \subset J(R) \ll R$ so $aR \ll R$. If $aR \ll R$, then $aR \subset J(R)$ and so $a \in aR \subset J(R)$. This shows that $a \in J(R)$ if and only if $aR \ll R$.

2.10.2 Definition. [1] Let *R* be a ring. An element $x \in R$ is called *right* (*left*) *quasi-regular* if 1 - x has a right (resp. left) inverse in *R*.

An element $x \in R$ is called *quasi-regular* if it is right and left quasi-regular.

A subset of *R* is said to be (*right*, *left*) *quasi-regular* if every element in it has the corresponding property.

2.10.3 Proposition. Given a ring R for each of the following subsets of R is equal to the radical J(R) of R.

- (J_1) The intersection of all maximal right (left) ideals of R;
- (J_2) The intersection of all right (left) primitive ideals of R;
- $(J_3) \{ x \in R \mid rxs is quasi-regular for all r, s \in R \};$
- $(J_4) \{ x \in R \mid rx \text{ is quasi-regular for all } r \in R \};$
- $(J_5) \{ x \in R \mid xs \text{ is quasi-regular for all } s \in R \};$

 (J_6) The union of all the quasi-regular right (left) ideals of R;

 (J_7) The union of all the quasi-regular ideals of R;

 (J_8) The unique largest superfluous right (left) ideals of R;

Moreover, (J_3) , (J_4) , (J_5) , (J_6) and (J_7) also describe the radical J(R) if "quasiregular" is replaced by "right quasi-regular" or by "left quasi-regular".

Proof. See [1, Theorem 15.3].

2.10.4 Proposition. Let R be a ring with radical J(R). Then for every right Rmodule M,

$$J(R)M_R \subset Rad(M_R).$$

If R is semisimple modulo its radical, then for every right R-module,

$$V(R)M_R = Rad(M_R)$$

and $M/J(R)M_R$ is semisimple.

Proof. See [1, Corollary 15.18].



CHAPTER 3 RESEARCH RESULT

In this chapter, we present the results of small principally injective modules and small principally injective injective rings.

3.1 SP-injective Modules

3.1.1 Definition. [12] Let R be a ring. A right R-module M is called *small* principally injective (briefly, SP-injective) if every R-homomorphism from a small and principal right ideal aR to M can be extended to an R-homomorphism from R to M.

3.1.2 Lemma. Let M be right R-modules. Then M is SP-injective if and only if for each $a \in J(R)$, $l_M r_R(a) = Ma$.

Proof. Clearly $Ma \subset l_M r_R(a)$. (\Rightarrow) Assume that M is SP-injective. Let $a \in J(R)$. To show that $l_M r_R(a) = Ma$. Let $x \in l_M r_R(a)$. Define $\varphi : aR \to xR$ by $\varphi(ar) = xr$, for every $r \in R$. To show that φ is the function. Let ar_1 and ar_2 be elements in aR such that $ar_1 = ar_2$. Then $ar_1 - ar_2 = 0$ and so $a(r_1 - r_2) = 0$. and $a(r_1 - r_2) = 0$, $x(r_1 - r_2) = 0$. Hence $xr_1 - xr_2 = 0$, then $xr_1 = xr_2$. Therefore $\varphi(ar_1) = xr_1 = xr_2 = \varphi(ar_2)$. This shows that φ is well-defined. Let ar_1 , $ar_2 \in aR$ and $r \in R$. Then $\varphi(ar_1r + ar_2) = \varphi(a(r_1r + r_2)) =$ $x(r_1r + r_2) = xr_1r + xr_2 = \varphi(ar_1)r + \varphi(ar_2)$. This shows that φ is an R-homomorphism. Since M is SP-injective, there exists an R-homomorphism $\hat{\varphi} : R \to M$ such that $\hat{\varphi} i_2 = i_1\varphi$ where i_1 : $xR \to M$ and i_2 : $aR \to R$ are the inclusion maps. Then $x = \varphi(a) = \hat{\varphi}(a) =$ $\hat{\varphi}(1.a) = \hat{\varphi}(1)a \in Ma$.

 (\Leftarrow) Let $a \in J(R)$, and let $\varphi: aR \to M$ be an *R*-homomorphism. Then $\varphi(a) \in l_M r_R(a)$, so by assumption, we have $\varphi(a) = xa$ for some $x \in M$. Define $\hat{\varphi}: R \to M$ by $\hat{\varphi}(r) = xr$ every $r \in R$. It is clear that $\hat{\varphi}$ is an *R*-homomorphism and is an extension of φ .

3.1.3 Example. Let
$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$$
 where F is a field, $M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then M is

SP-injective.

Proof. We have only $A_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $A_3 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$, $A_5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $A_6 = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ are right ideal of *R*, and we see that only $A_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is only the nonzero small principal right ideal of R because for every $A_i \subset R$, $2 \le i \le 5$, $A_i \ne R$ then $A_1+A_i \neq R$. Since, for each $x = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = A_1, \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, i.e., $xR = A_1$, A_1 is a principal right ideal of R. Let $\varphi: A_1 \rightarrow M$ be an R-homomorphism. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in A_1$, there exists $x_{11}, x_{12} \in F$ such that $\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$. Then $\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$. $\varphi\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right) = \varphi\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right) = \begin{pmatrix}x_{11} & x_{12}\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & x_{12}\\ 0 & 0\end{pmatrix}.$ Then $\begin{pmatrix}x_{11} & x_{12}\\ 0 & 0\end{pmatrix} = \begin{pmatrix}x_{11} & x_{12}\\ 0 & 0\end{pmatrix}$ $\begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}$ so $x_{11} = 0$. Define $\hat{\varphi} : R \to M$ by $\hat{\varphi} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} x_{12}a_{11} & x_{12}a_{12} \\ 0 & 0 \end{pmatrix}$ for every $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in R$. To show that $\hat{\varphi}$ is well-defined. Let $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$, $\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} \in R$ such that $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}. \text{ Then } \hat{\varphi} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} x_{12}a_{11} & x_{12}a_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{12}b_{11} & x_{12}b_{12} \\ 0 & 0 \end{pmatrix} =$ $\hat{\varphi}\left(\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} \right)$. To show that $\hat{\varphi}$ is an *R*-homomorphism. Let $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$, $\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}$ $\begin{pmatrix} F & F \\ 0 & F \end{pmatrix} \text{ and } \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \in R. \text{ Then } \hat{\varphi} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} + \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{12}$ $\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11}r_1 + b_{11} & a_{11}r_2 + a_{12}r_3 + b_{12} \\ 0 & a_{22}r_3 + b_{22} \end{pmatrix} = \begin{pmatrix} x_{12}(a_{11}r_1 + b_{11}) & x_{12}(a_{22}r_3 + b_{22}) \\ 0 & 0 \end{pmatrix} =$ $\begin{pmatrix} x_{12}a_{11}r_1 + x_{12}b_{11} & x_{12}a_{22}r_3 + x_{12}b_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{12}a_{11}r_1 & x_{12}a_{22}r_3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} x_{12}b_{11} & x_{12}b_{22} \\ 0 & 0 \end{pmatrix}$ $\hat{\varphi} \begin{pmatrix} a_{11}r_1 & a_{11}r_2 + a_{12}r_3 \\ 0 & a_{22}r_3 \end{pmatrix} + \hat{\varphi} \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} = \hat{\varphi} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} + \hat{\varphi} \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix} =$

$$\hat{\varphi}\left(\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix}\right)\begin{pmatrix}r_{1}&r_{2}\\0&r_{3}\end{pmatrix} + \hat{\varphi}\left(\begin{pmatrix}b_{11}&b_{12}\\0&b_{22}\end{pmatrix}\right). \text{ To show that } \hat{\varphi}\iota = \varphi. \text{ Let } \begin{pmatrix}0&x\\0&0\end{pmatrix} \in A_{1}. \text{ Then } \hat{\varphi}\iota\left(\begin{pmatrix}0&x\\0&0\end{pmatrix}\right) = \hat{\varphi}\left(\iota\begin{pmatrix}0&x\\0&0\end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix}0&x\\0&0\end{pmatrix}\right) = \begin{pmatrix}0&x_{12}x\\0&0\end{pmatrix} = \begin{pmatrix}x_{11}&x_{12}\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&x\end{pmatrix} = \varphi\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&x\end{pmatrix}\right) = \varphi\left(\begin{pmatrix}0&x\\0&0\end{pmatrix}\right). \text{ This shows that } \hat{\varphi} \text{ is an extension of } \varphi. \text{ Thus } M \text{ is } SP\text{-injective.}$$

3.1.4 Proposition. Let M be $\{M_i, i \in I\}$ be a family of right R-modules. Then the direct product $\prod_{i \in I} M_i$ is SP-injective if and only if each M_i is SP-injective.

Proof. (\Rightarrow) Let { $M_i, i \in I$ } be a family of right *R*-modules and the direct product $\prod_{i \in I} M_i$ is *SP*-injective. Let $i \in I$, we must show that M_i is *SP*-injective. Let $a \in R$, $aR \ll R$ and let $\varphi : aR \to M_i$ be an *R*-homomorphism. Let π_i and φ_i , for each $i \in I$, be the *i*-th projection map and the *i*-th injection map, respectively. Since $\prod_{i \in I} M_i$ is *SP*-injective, there exists an *R*-homomorphism $\hat{\varphi} : R \to \prod_{i \in I} M_i$ such that $\hat{\varphi}_i = \varphi_i \varphi$ where $\iota : aR \to R$ is the inclusion map. Thus $\pi_i \hat{\varphi} \iota = \pi_i \varphi_i \varphi$, so by Definition 2.6.2, $\pi_i \hat{\varphi} \iota = \varphi$. Thus $\pi_i \hat{\varphi}$ is an extension of φ .

 (\Leftarrow) Let M_i is SP – injective. Let $a \in R$, $aR \ll R$ and let $\varphi : aR \to \prod_{i \in I} M_i$ be an R-homomorphism. Let π_i be the *i*-th projection map. Since, for each *i*, M_i is SP-injective, there exists an R-homomorphism $\alpha_i : R \to M_i$ such that $\pi_i \varphi = \alpha_i \iota$ where $\iota : aR \to R$ is the inclusion map. Then by Definition 2.6.5 and Proposition 2.6.6, we obtain $\hat{\varphi} : M \to \prod_{i \in I} N_i$ such that $\pi_i \hat{\varphi} = \alpha_i$ for each $i \in I$. Then $\pi_i \hat{\varphi} \iota = \alpha_i \iota$, so $\pi_i \varphi = \alpha_i \iota = \pi_i \hat{\varphi} \iota$.

3.1.5 Lemma. Let M_i $(1 \le i \le n)$ be SP-injective modules. Then $\underset{i=1}{\overset{n}{\oplus}}M_i$ is SP-injective if and only if each M_i is SP-injective.

Proof. (\Rightarrow) Assume that M_i ($1 \le i \le n$) be *R*-modules and $\bigcap_{i=1}^n M_i$ is *SP*-injective. Let $i \in I$, we must show that M_i is *SP*-injective. Let $a \in R$, $aR \ll R$ and let $\varphi : aR \to M_i$ be an *R*-homomorphism. Let π_i and φ_i for each $i \in I$, be the *i*-th projection map and the *i*-th injection map, respectively. Since $\bigcap_{i=1}^n M_i$ is *SP*-injective, there exists an *R*-homomorphism $\hat{\varphi} : R \to \bigcap_{i=1}^n M_i$ such that $\hat{\varphi} \iota = \varphi_i \varphi$ where $\iota : aR \to R$ is the inclusion map. Thus $\pi_i \hat{\varphi} \iota = \pi_i \varphi_i \varphi$, so by Definition 2.6.2, $\pi_i \hat{\varphi} \iota = \varphi$. Thus $\pi_i \hat{\varphi}$ is an extension of φ .

 $(\Leftarrow) \text{ Let } a \in J(R) \text{ and } \varphi : aR \to \bigcap_{i=1}^{n} M_{i} \text{ be an } R\text{-homomorphism. Since for each } i \in \{1, 2, 3, \dots, n\}, M_{i} \text{ is } SP\text{-injective, there exists an } R\text{-homomorphism } \varphi_{i} : R \to M_{i} \text{ such that } \varphi_{i} : i = \pi_{i}\varphi \text{ where } \pi_{i} \text{ is the } i\text{-th projection map from } \bigcap_{i=1}^{n} M_{i} \text{ to } M_{i} \text{ and } i : aR \to R \text{ is the inclusion map. Set } \hat{\varphi} = i_{1}\varphi_{l} + i_{2}\varphi_{2} + \ldots + i_{n}\varphi_{in} : R \to \bigcap_{i=1}^{n} M_{i} \text{ where } i_{i} : M_{i} \to \bigcap_{i=1}^{n} M_{i}$ for each $i \in \{1, 2, 3, \dots, n\}$ is the *i*-injection map. We must show that $\hat{\varphi}$ is an extension of φ . Let $a(r) \in s(R)$. Then $\hat{\varphi}_{l}(a(r)) = \hat{\varphi}(a(r)) = i_{1}\varphi_{1}(a(r)) + i_{2}\varphi_{2}(a(r)) + \ldots + i_{n}\varphi_{n}(a(r)) = \varphi_{1}(a(r)) + \varphi_{2}(a(r)) + \ldots + \varphi_{n}(a(r)) = \varphi_{1}i_{1}(a(r)) + \varphi_{2}i_{2}(a(r)) + \ldots + \varphi_{n}i_{n}(a(r)) = \pi_{1}\varphi(a(r)) + \pi_{2}\varphi(a(r)) + \ldots + \pi_{n}\varphi(a(r)) = (\pi_{1}+\pi_{2}+\ldots+\pi_{n})\varphi(a(r)) = \varphi(a(r)).$ Then $\bigcap_{i=1}^{n} M_{i}$ is SP-injective.

3.1.6 Lemma. Any direct summand of SP-injective module is again SP-injective.

Proof. Let *M* be an *SP*-injective module and let *A* be a direct summand of *M*. To show that *A* is an *SP*-injective. Let $a \in R$, $aR \ll R$ and let $\varphi: aR \to A$ be an *R*homomorphism. Since *M* is *SP*-injective, there exists an *R*-homomorphism $\hat{\varphi}: R \to M$ such that $\alpha \varphi = \hat{\varphi} \iota$ where $\iota: aR \to R$ is the inclusion map and $\alpha: A \to M$ is the injection map. Let $\pi: M \to A$ be the projection map. Then $\pi \alpha \varphi = \pi \hat{\varphi} \iota$. Hence by Definition 2.6.2, $\varphi = \pi \hat{\varphi} \iota$. Then $\pi \hat{\varphi}$ is an extension of φ .

3.2 SP - injective Rings

If R_R is an SP-injective modules, then we call R is a right SP-injective ring. In this section, we give some properties and characterizations of SP-injective rings.

3.2.1 Lemma. [12] *The following conditions are equivalent for a ring R*

- (1) *R* is right SP-injective ring.
- (2) lr(a) = Ra for any $a \in J(R)$.
- (3) $r(a) \subset r(b)$, where $a \in J(R)$, $b \in R$ implies $Rb \subset Ra$.
- (4) $l(r(a) \cap bR) = l(b) + Ra$ for all $a \in J(R)$, $b \in R$.
- (5) If $\alpha : aR \rightarrow R$, $a \in J(R)$, is an *R*-homomorphism, then $\alpha(a) \in Ra$.

3.2.2 Theorem. Let *R* be a right *SP-injective* ring. Then

(1) *lr*(*Ra*) = *Ra*, for any *a* ∈ *J*(*R*).
(2) If *aR* ⊕ *bR* and *Ra* ⊕ *Rb* are both direct, *a,b* ∈ *J*(*R*), then *l*(*a*)+*l*(*b*) = *R*.

Proof. (1) Let *R* be a right *SP*-injective ring and let $a \in J(R)$. To show that $lr(Ra) = Ra.(\Box)$ Let $ra \in Ra$. To show that $ra \in lr(Ra)$. Let $s \in R$, and Ras = 0. Then ras = 0 and hence $ra \in lr(Ra)$. (\Box) Let $x \in lr(Ra)$. Define $\varphi: aR \to xR$ by $\varphi(ar) = xr$, for every $r \in R$. To show that φ is the function. Let ar = 0 then $\varphi(ar) = xr = 0$. This shows that φ is well-defined. Let ar_1 , $ar_2 \in aR$ and $r \in R$. Then $\varphi(ar_1r + ar_2) = \varphi(a(r_1r + r_2)) = x(r_1r + r_2) = xr_1r + xr_2 = \varphi(ar_1)r + \varphi(ar_2)$. This shows that φ is an *R*-homomorphism. Since *R* is a right *SP*-injective ring. Then there exists $\hat{\varphi}: R \to R$ an *R*-homomorphism, such that $i_1\varphi = \hat{\varphi}(a) = \hat{\varphi}(1)a \in Ra$.

(2) Let *R* be a right *SP*-injective ring, $a, b \in J(R)$ and let $aR \oplus bR$ and $Ra \oplus Rb$ are both direct. To show that l(a)+l(b) = R. Define $\varphi: (a+b)R \to R$ by $\varphi(a+b)r = br$, for every $r \in R$. To show that φ is the function. If (a+b) = 0, then $ar = br \in aR \cap bR = 0$ so br = 0. Then $\varphi(a+b)r = br = 0$. This shows that φ is well-defined. We now show that φ is an *R*-homomorphism. Let $(a+b)r_1$, $(a+b)r_2 \in (a+b)R$ and $r \in R$. Then $\varphi((a+b)r_1r + (a+b)r_2) = \varphi((a+b)(r_1r + r_2)) = b(r_1r + r_2) = br_1r + br_2 = \varphi(a+b)r_1r + \varphi(a+b)r_2$. This shows that φ is an *R*-homomorphism. Since *R* is a right *SP*-injective, there exists an *R*-homomorphism $\hat{\varphi} : R \to R$ such that $\varphi = \hat{\varphi} i$ where $i : (a+b)R \to R$ is the inclusion map. Hence $\hat{\varphi}(1)(a+b) = \hat{\varphi}(1.(a+b)) = \hat{\varphi}(a+b) = \varphi(a+b) = b$ so $\hat{\varphi}(1)$ (a+b) = b. Then $\hat{\varphi}(1)a + \hat{\varphi}(1)b = b$, and so $\hat{\varphi}(1)a = b - \hat{\varphi}(1)b = (1 - \hat{\varphi}(1))b \in Ra \cap Rb$ = 0. Then $\hat{\varphi}(1) \in l(a)$ and $(1 - \hat{\varphi}(1)) \in l(b)$. Hence $1 = \hat{\varphi}(1) + (1 - \hat{\varphi}(1)) \in l(a) + l(b)$. Then $1 \in l(a) + l(b)$ so l(a) + l(b) = R.

3.2.3 Proposition. If R is a right SP-injective, so is eRe for all $e^2 = e \in R$ satisfying ReR = R.

Proof. Let *R* be a right *SP*-injective and *e* be an idempotent sastisfying ReR = R. Write S = eRe. Let $a \in J(eSe)$ and let $\varphi : aS \to S$ be an *S*-homomorphism. To show that $r(a) \subset r(\varphi(a))$. Let $x \in r(a)$. Then ax = 0. Hence $\varphi(a)x = \varphi(ax) = \varphi(0) = 0$. This shows that $r(a) \subset r(\varphi(a))$, so $lr\varphi(a) \subset lr(a)$ by proposition 2.3.2 (3). Since a(eRe)R = ae(ReR) = aeR = aR. Since $aSR \subset JR \ll R$, $aSR \ll R$, so $aR \ll R$. Then by Lemma 3.1, lr(a) = Ra. It follows that $R\varphi(a) \subset lr(\varphi(a)) \subset lr(a) = Ra$. Then $\varphi(a) = e\varphi(a)$. Since $\varphi(a) = 1_R\varphi(a) \in R\varphi(a) \subset Ra$, $\varphi(a) \in Ra$ so $e\varphi(a) \in eRa$. Then $\varphi(a) = e\varphi(a)$. Since $\varphi(a) = 1_R\varphi(a) \in R\varphi(a) \subset Ra$, $\varphi(a) \in Ra$ so $e\varphi(a) \in eRa$. Then $\varphi(a) = e\varphi(a)$. Since $\varphi(a) = 1_R\varphi(a) \in \varphi(a) = sa$ for some $s \in S$. Define $\hat{\varphi} : S \to S$ by $\hat{\varphi}(t) = st$ for every $t \in S$. Let $t_1, t_2 \in S$ such that $t_1 = t_2$. Then $st_1 = st_2$. Hence $\hat{\varphi}(t_1) = st_1 = st_2 = \hat{\varphi}(t_2)$. This shows that $\hat{\varphi}$ is well-defined. Let $t_1, t_2 \in S$ and $t \in S$. Then $\hat{\varphi}(t_1t+t_2) = s(t_1t+t_2) = st_1t+st_2 = \hat{\varphi}(t_1)t + \hat{\varphi}(t_2)$. This shows that $\hat{\varphi}$ is *S*-homomorphism. To show that $\varphi = \hat{\varphi}i$. Let $at \in aS$. Then $\varphi(at) = \varphi(a)t = sat = \hat{\varphi}(a)t = \hat{\varphi}(at) = \hat{\varphi}i(at)$. Hence eRe is right *SP*-injective.

3.2.4 Theorem. Let *R* be right *SP*-injective, $a \in R$ and $b \in J(R)$.

- (1) If bR embeds in aR, then Rb is an image of Ra.
- (2) If aR is an image of bR, then Ra embeds in Rb.
- (3) If $bR \cong aR$, then $Ra \cong Rb$.

Proof. (1) Let $f : bR \to aR$ be an *R*-monomorphism. Since *R* is right *SP*-injective, there exists an *R*-homomorphism $\hat{f} : R \to R$ such that $\iota_2 f = \hat{f}\iota_1$ where $\iota_1 : bR \to R$ and $\iota_2 : aR \to R$ are the inclusion maps. Define $\sigma : Ra \to Rb$ by $\sigma(sa) = s\hat{f}(b)$ for every $s \in R$. If sa = 0, then $\sigma(sa) = s\hat{f}(b) = sf(b) \in s(aR) =$ (sa)R = 0. To show that $Im(\hat{f}b) \subset Im(a)$. This shows that σ is well-defined. To show that σ is a left *R*-homomorphism. Let $s_1(a), s_2(a) \in Ra$ and $v \in R$. Then $\sigma(vs_1a + s_2a) =$ $\sigma((vs_1 + s_2)a) = (vs_1+s_2)\hat{f}b = vs_1\hat{f}b + s_2\hat{f}b = v(s_1\hat{f}b) + s_2\hat{f}b = v\sigma(s_1a) + \sigma(s_2a)$. To show that σ is an *R*-epimorphism. Let $kb \in Rb$. To show that $r(\hat{f}(b)) \subset r(b)$. Let $x \in$ $r(\hat{f}(b)$. Then $\hat{f}(b(x)) = 0$, so $f(b(x)) = \hat{f}(b(x)) = 0$. Since *f* is monic, bx = 0. Then $x \in$ r(b) and hence $lr(b) \subset lr(\hat{f}(b))$. Since $bR \ll R$ and $\hat{f} : R \to R$ is an *R*-homomorphism, $\hat{f}(b)R \ll R$ by Proposition 2.2.4. Since *R* is *SP*-injective, $Rb \subset R\hat{f}b$ by Lemma 3.2.1. Then $b=1\cdot b=s\hat{f}b$ for some $s \in R$. Hence there exists $ksa \in Ra$ such that $kb = \sigma(ksa)$.

(2) Let $f: bR \to aR$ be an *R*-epimorphism. Since *R* is *SP*-injective, there exists an *R*-homomorphism $\hat{f}: R \to R$ such that $\iota_2 f = \hat{f}\iota_1$ where $\iota_1: bR \to R$ and $\iota_2: aR \to R$ are the inclusion maps. Define $\sigma: Ra \to Rb$ by $\sigma(sa) = s\hat{f}(bx)$ for every $s \in R$. It is clear that σ is a left *R*-homomorphism. Let $sa \in Ker(\sigma)$. Then $0 = \sigma(sa) = s\hat{f}(bx) = sf(bx) = sa = 0$.

(3) Follows from (1) and (2) \Box

Following[1], a ring is *R* semiprimitive in case J(R) = 0.

3.2.5 Proposition. The following conditions are equivalent for a ring *R*:

- (1) *R* is semiprimitive.
- (2) Every right *R*-module is SP-injective.
- (3) Every principal right ideal is SP-injective.

Proof. We only prove the right side, the left side is analogously. It is ovious that (1) \Rightarrow (2) \Rightarrow (3). We show (3) \Rightarrow (1). Suppose $J \neq 0$. Then there exists a nonzero element $a \in J(R)$. Then by assumption, the inclusion map from aR to R is split. Then aR is direct summand of R so aR = 0 which is contradiction.

3.2.6 Theorem. The following conditions are equivalent for a ring *R*:

- (1) Every small and principal right ideal of *R* is *projective*.
- (2) Every factor module of an SP-injective module is SP-injective.
- (3) Every factor module of an injective *R*-module is *SP-injective*.

Proof. (1) \Rightarrow (2) Let *M* be an *SP*-injective module, *X* a submodule of *M*. To show that *M*/*X* is an *SP*-injective. Let $a \in J(R)$ and let $\varphi : aR \rightarrow M/X$ be an *R*-homomorphism. Since *aR* is projective, there exists an *R*-homomorphism $\alpha : aR \rightarrow M$ such that $\varphi = \eta \alpha$ where $\eta : M \rightarrow M/X$ is the natural *R*-epimorphism. Since *M* is *SP*-injective, there exists an *R*-homomorphism $\beta : R \rightarrow M$ such that $\alpha = \beta \iota$ where $\iota : aR \rightarrow R$ is the inclusion map. Then $\varphi = \eta \alpha = \eta \beta \iota$. Therefore $\eta \beta$ is an extension of φ . Thus *M*/*X* is an *SP*-injective.

(2) \Rightarrow (3) Let *M* be an injective *R*-module and *X* be a submodule of *M*. It is clear that an injective *R*-module is an *SP*-injective module, so *M* is *SP*-injective. Then by (2), *M*/*X* is an *SP*-injective.

(3) \Rightarrow (1) Let $aR \ll R$, $\gamma : A \to B$ be an *R*-epimorphism and let $\varphi : aR \to B$ be an *R*-homomorphism. Let *E* be an injective *R*-module and embed *A* in *E* by Proposition 2.5.4. Since γ is an *R*-epimorphism, by Proposition 2.4.4, there exists an *R*-isomorphism σ : $A/Ker(\gamma) \to B$ such that $\gamma = \sigma \eta_1$ where $\eta_1 : A \to A/Ker(\gamma)$ is the natural *R*-epimorphism. Then by Proposition 2.1.15, we have $\sigma^{-1} : B \to A/Ker(\gamma)$ is an *R*-isomorphism, so $B \cong A/Ker(\gamma)$ and $A/Ker(\gamma)$ is a submodule of $E/Ker(\gamma)$. By

assumption, there exists an *R*-homomorphism $\hat{\varphi}: M \rightarrow E/Ker(\gamma)$ such that $\iota_1 \sigma^{-1} \varphi$ $\hat{\varphi}_{l_2}$ where $\iota_1: A/Ker(\gamma) \to E/Ker(\gamma)$ and $\iota_2: aR \to R$ are the inclusion maps. Since R is projective, there exists an R-homomorphism $\beta: R \to E$ such that $\hat{\varphi} = \eta_2 \beta$ where $\eta_2 : E \to E/Ker(\gamma)$ is the natural *R*-epimorphism. Then $\hat{\varphi}_{l_2} = \eta_2 \beta_{l_2}$. Hence $\iota_1 \sigma^{-1} \varphi$ $=\hat{\varphi}\iota_2 = \eta_2\beta\iota_2$. It follows that $\iota_1\sigma^{-1}\varphi = \eta_2\beta\iota_2$. To show that $\beta(aR) \subset A$. Let a(r) $\in a(R)$. Then $\iota_1 \sigma^{-1} \varphi(a(r)) = \eta_2 \beta \iota_2(a(r)) = \eta_2 \beta(a(r)) = \eta_2(\beta(a(r))) = \beta(a(r)) + \beta(a(r)) = \beta(a(r)) = \beta(a(r)) + \beta(a(r)) = \beta(a(r)) = \beta(a(r)) = \beta(a(r)) + \beta(a(r)) = \beta$ $Ker(\gamma)$. Hence $\iota_1 \sigma^{-1} \varphi(a(r)) = \sigma^{-1} \varphi(a(r)) = a + Ker(\gamma)$ for some $a \in A$, so $\beta(a(r)) + \beta(a(r)) = a + Ker(\gamma)$ for some $a \in A$, so $\beta(a(r)) + \beta(a(r)) = a + Ker(\gamma)$ for some $a \in A$, so $\beta(a(r)) = a + Ker(\gamma)$ for some $a \in A$, so $\beta(a(r)) = a + Ker(\gamma)$ for some $a \in A$, so $\beta(a(r)) = a + Ker(\gamma)$ for some $a \in A$. $Ker(\gamma) = a + Ker(\gamma)$. Thus $\beta(a(r)) - a \in Ker(\gamma)$. It follows that $\beta(a(r)) = (\beta(a(r)) - \beta(a(r)))$ a) + $a \in Ker(\gamma)$ + A = A. To show that $\varphi = \gamma \beta$. Let $a(r) \in a(R)$. Then $\iota_1 \sigma^{-1} \varphi(a(r)) = i \sigma^{-1} \varphi(a(r))$ $\sigma^{-1} \varphi(a(r)) = \eta_2 \beta \iota_2(a(r)) = \eta_2 \beta(a(r))$. Hence $\iota_1 \sigma^{-1} \varphi(a(r)) = \eta_2 \beta(a(r)) = \beta(a(r)) + \beta(a(r)) = \beta(a(r)) = \beta(a(r)) + \beta(a(r)) = \beta(a(r)) = \beta(a(r)) + \beta(a(r)) = \beta(a(r)) = \beta(a(r)) = \beta(a(r)) + \beta(a(r)) = \beta(a$ $Ker(\gamma)$, so $\iota_1 \sigma^{-1} \varphi(a(r)) = \beta(a(r)) + Ker(\gamma)$. Since γ is an *R*-epimorphism, $\varphi(a(r)) =$ $\gamma(a)$ for some $a \in A$. Thus $\iota_1 \sigma^{-1} \varphi(a(r)) = \iota_1 \sigma^{-1} \gamma(a) = \sigma^{-1} \gamma(a) = \eta_1(a) = a + \eta_1(a)$ $Ker(\gamma)$. It follows that $\beta(a(r)) + Ker(\gamma) = a + Ker(\gamma)$. Then $\beta(a(r)) - a \in Ker(\gamma)$. Hence $\gamma(\beta(a(r)) - a) = 0$, so $\gamma\beta(a(r)) = \gamma(a) = \varphi(a(r))$. Thus $\gamma\beta(a(r)) = \varphi(a(r))$. This shows that β lifts φ .

3.2.7 Proposition. Let *R* be right *SP-injective* and $b_i \in J(R)$, $(1 \le i \le n)$.

(1) If $Rb_1 \oplus ... \oplus Rb_n$ is direct, then any *R*-homomorphism $\alpha : b_1R + ... + b_nR \rightarrow R$ can be extended to *R*.

(2) If
$$b_1 R \oplus \ldots \oplus b_n R$$
 is direct, then $R(b_1 + \ldots + b_n) = Rb_1 + \ldots + Rb_n$.

Proof. (1) Let $Rb_1 \oplus ... \oplus Rb_n$ is direct and let $\alpha : b_1R + ... + b_nR \to R$ be an Rhomomorphism. Since R is SP-injective, for each $i, 1 \le i \le n$, there exists an Rhomomorphism $\varphi_i : R \to R$ such that $\alpha(b_i r) = \varphi_i(b_i r)$ for every $r \in R$. Since $b_i(R) \ll R$ for each i = 1, 2, ..., n, $\sum_{i=1}^n b_i(R) \ll R$ by Proposition 2.2.3(2), and we have $(\sum_{i=1}^n b_i)(R) \subset \sum_{i=1}^n b_i(R)$ which implies $(\sum_{i=1}^n b_i)(R) \ll R$ by Proposition 2.2.3(1).
Since R is SP-injective, there exists an R-homomorphism $\varphi : R \to R$ such that, for any $r \in R, \varphi(\sum_{i=1}^n b_i)(r) = \alpha(\sum_{i=1}^n b_i)(r)$. To show that $\sum_{i=1}^n \varphi(b_i) = \sum_{i=1}^n \varphi_i(b_i)$. Let $r \in R$.

Then $\sum_{i=1}^{n} \varphi_i b_i(r) = \varphi_1 b_1(r) + \varphi_2 b_2(r) + \dots + \varphi_n b_n(r) = \alpha b_1(r) + \alpha b_2(r) + \dots + \alpha b_n(r)$ $= (\alpha b_1 + \alpha b_2 + \dots + \alpha b_n)(r) = \alpha (b_1 + b_2 + \dots + b_n)(r) = \alpha (\sum_{i=1}^{n} b_i)(R) = \varphi (\sum_{i=1}^{n} b_i)(R)$ $= \varphi (b_1 + b_2 + \dots + b_n)(r) = (\varphi b_1 + \varphi b_2 + \dots + \varphi b_n)(r) = \varphi b_1(r) + \varphi b_2(r) + \dots + \varphi b_n(r) =$ $\sum_{i=1}^{n} \varphi b_i(r)$ This shows that $\sum_{i=1}^{n} \varphi (b_i) = \sum_{i=1}^{n} \varphi_i(b_i)$. Then $(\varphi_1 b_1 - \varphi b_1) + (\varphi_2 b_2 - \varphi b_2) + \dots + (\varphi_n b_n - \varphi b_n) = 0$. Thus $(\varphi_1 - \varphi)b_1 + (\varphi_2 - \varphi)b_2 + \dots + (\varphi_n - \varphi)b_n = 0$. Since $Rb_1 \oplus Rb_2 \oplus \dots \oplus Rb_n$ is direct, $(\varphi_1 - \varphi) = (\varphi_2 - \varphi) = (\varphi_n - \varphi) = 0$. Then by Proposition 2.6.8, $(\varphi_1 - \varphi)b_1 = (\varphi_2 - \varphi)b_2 = \dots = (\varphi_n - \varphi)b_n = 0$. Hence $(\varphi_i - \varphi)b_i =$ 0, for all $1 \le i \le n$. Thus $\varphi_i(b_i) = \varphi(b_i)$, for all $1 \le i \le n$. To show that $\alpha = \varphi i$. Let $b_1(x_1) + b_2(x_2) + \dots + b_n(x_n) \in b_1(R) + b_2(R) + \dots + b_n(R)$. Then $\alpha(b_1(x_1) + b_2(x_2) + \dots + b_n(x_n)) = \alpha b_1(x_1) + \alpha b_2(x_2) + \dots + \alpha b_n(x_n) = \varphi(b_1(x_1) + b_2(x_2) + \dots + \varphi b_n(x_n)) = \varphi(b_1(x_1) + b_2(x_2) + \dots + \beta b_n(x_n)) = \varphi(b_1(x_1) + b_2(x_2) + \dots + b_n(x_n))$. Hence $\alpha(b_1(x_1) + b_2(x_2) + \dots + b_n(x_n)) = \varphi_i(b_1(x_1) + b_2(x_2) + \dots + b_n(x_n))$. This shows that φ is an extension of α .

(2) (\supset) Let $\alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n \in Rb_1 + Rb_2 + \ldots + Rb_n$. To show that $\alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n \in R(b_1 + b_2 + \ldots + b_n)$. For each *i*, define $\varphi_i : (b_1 + b_2 + \ldots + b_n)R \to R$ by $\varphi_i((b_1 + b_2 + \ldots + b_n)r) = b_ir$ for every $r \in R$. Let $0 = (b_1 + b_2 + \ldots + b_n)R \to R$ by $\varphi_i((b_1 + b_2 + \ldots + b_n)R$. Then $b_1(r) + b_2(r) + \ldots + b_n(r) = (b_1 + b_2 + \ldots + b_n)R = 0$. Since $b_1R \oplus b_2R \oplus \ldots \oplus b_nR$ is direct, $b_1r = b_2r = \ldots = b_nr = 0$ so $b_ir = 0$. This shows that φ_i is well-defined. Let $(b_1 + b_2 + \ldots + b_n)r_1, (b_1 + b_2 + \ldots + b_n)r_2 \in (b_1 + b_2 + \ldots + b_n)(r_1)r + (b_1 + b_2 + \ldots + b_n)(r_2)) = \varphi_i((b_1 + b_2 + \ldots + b_n)(r_1r + r_2)) = b_i(r_1r + r_2) = b_i(r_1r) + b_i(r_2) = b_i(r_1)r + b_i(r_2) = \varphi_i((b_1 + b_2 + \ldots + b_n)(r_1))r + \varphi_i((b_1 + b_2 + \ldots + b_n)(r_2))$. This shows that φ_i is an *R*-homomorphism. By the similar proof of (1) we have $(b_1 + b_2 + \ldots + b_n)R \ll R$. Since *R* is *SP*-injective, there exists an *R*-homomorphism $\hat{\varphi}_i : R \to R$ such that $\varphi_i = \hat{\varphi}_i \iota$ where $\iota : (b_1 + b_2 + \ldots + b_n)R$ $\rightarrow R$ is the inclusion map. Then $b_i = \varphi_i(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2 + \ldots + b_n)$ so $\alpha_1b_1 + \alpha_2b_2 + \ldots + \alpha_nb_n = \alpha_1\hat{\varphi}_i(b_1 + b_2 + \ldots + b_n) + \alpha_2\hat{\varphi}_2(b_1 + b_2 + \ldots + b_n) + \ldots + \alpha_n\hat{\varphi}_n(b_1 + b_2 + \ldots + b_n) = (\alpha_1\hat{\varphi}_1 + \alpha_2\hat{\varphi}_2 + \ldots + \alpha_n\hat{\varphi}_n)(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2 + \ldots + b_n) + \ldots + \alpha_n\hat{\varphi}_n(b_1 + b_2 + \ldots + b_n) = (\alpha_1\hat{\varphi}_1 + \alpha_2\hat{\varphi}_2 + \ldots + \alpha_n\hat{\varphi}_n)(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2 + \ldots + b_n) + \ldots + \alpha_n\hat{\varphi}_n(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2 + \ldots + b_n) + \ldots + \alpha_n\hat{\varphi}_n(b_1 + b_2 + \ldots + b_n) \in R(b_1 + b_2$... + b_n). (\subset) Let $\alpha(b_1 + b_2 + ... + b_n) \in R(b_1 + b_2 + ... + b_n)$. Then $\alpha(b_1 + b_2 + ... + b_n) = \alpha b_1 + \alpha b_2 + ... + \alpha b_n \in Rb_1 + ... + Rb_n$.

3.2.8 Proposition. Let *R* be right *SP-injective* and $B_1 \oplus ... \oplus B_n$ a direct sum of small(two – side) ideals of *R*. Then for any fully invariant ideal *A* of *R*, we have

$$A \cap (B_1 \oplus \ldots \oplus B_n) = (A \cap B_1) \oplus \ldots \oplus (A \cap B_n).$$

Proof. (\supset) Since $A \cap B_i \subset A \cap (B_1 \oplus ... \oplus B_n)$ for each i = 1, 2, ..., n, we have $(A \cap B_1) \oplus ... \oplus (A \cap B_n) \subset A \cap (B_1 \oplus ... \oplus B_n)$. (\subset) Let $a = \sum_{i=1}^n b_i \in A \cap$ $(B_1 \oplus ... \oplus B_n)$. To show that $\sum_{i=1}^n b_i \in (A \cap B_1) \oplus ... \oplus (A \cap B_n)$. Let $\pi_k :$ $\bigoplus_{i=1}^n b_i R \to b_k R$ be the projection map. Since for each i, $(1 \le i \le n)$, $Rb_i \subset B_i$. Thus $\bigoplus_{i=1}^n Rb_i$ is direct. By Proposition 3.2.7, π_k has an extension $\hat{\pi}_k : R \to b_k R$ such that π_k $= \hat{\pi}_k \iota$ where $\iota : \bigoplus_{i=1}^n b_i R \to R$ is the inclusion map. Let $r_i \in R$. Then $b_i = \pi_i \sum_{i=1}^n b_i =$ $\hat{\pi}_i \iota \sum_{i=1}^n b_i = \hat{\pi}_i (\sum_{i=1}^n b_i) = \hat{\pi}_i (a) \in A \cap B_i$. Hence $\sum_{i=1}^n b_i = b_1 + b_2 + ... + b_n \in$ $A \cap B_1 \oplus A \cap B_2 \oplus ... \oplus A \cap B_n$.



Lists of References

- F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules," Graduate Texts in Math. No.13 ,Springer-verlag, New York, 1992.
- [2] V. Camillo, "Commutative Rings whose Principal Ideals are Annihilators," Portugal Math., Vol 46, 1989. pp 33-37.
- [3] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, "Extending Modules," Pitman, London, 1994.
- [4] A. Facchini, "Module Theory," Birkhauser Verlag, Basel, Boston, Berlin, 1998.
- [5] T.Y. Lam, "A First Course in Noncommutative Rings," Graduate Texts in Mathematics Vol 131, Springer-Verlag, New York, 1991.
- [6] S. H. Mohamed and B. J. Muller, "Continuous and Discrete Modules," London Math. Soc. Lecture Note Series 14, Cambridge Univ. Press, 1990.
- [7] W. K. Nicholson and M. F. Yousif, "Principally Injective Rings," J. Algebra, Vol 174, 1995. pp 77-93.
- [8] W. K. Nicholson and M. F. Yousif, "Mininjective Rings," J. Algebra, Vol 187, 1997. pp 548-578.
- [9] W. K. Nicholson, J. K. Park and M. F. Yousif, "Principally Quasi-injective Modules," Comm. Algebra, 27:4(1999). pp 1683-1693.
- [10] N. V. Sanh, K. P. Shum, S. Dhompongsa and S.Wongwai, "On Quasi-principally Injective Modules," Algebra Coll.6: 3, 1999. pp 269-276.
- [11] L. Shen and J. Shen, "Small Injective Rings," arXiv: Math., RA/0505445 vol 1, 2005.
- [12] L.V. Thuyet, and T.C.Quynh, "On Small Injective Rings, Simple-injective and Quasi-Frobenius Rings," Acta Math. Univ. Comenianae, Vol 78(2), 2009. pp 161-172.
- [13] R. Wisbauer, "Foundations of Module and Ring Theory," Gordon and Breach Science Publisher, 1991.
- [14] P.B. Bhattacharya, S.K. Jain and S.R. Nagpaul, "Basic Abstract Algebra," The Press Syndicate of the University of Cambridge, second edition, 1995.

Lists of References (Continued)

- [15] B. Hartley and T. O. Hawkes, "Ring, Modules and Linear Algebra," University Press, Cambridge, 1983.
- [16] S. Wongwai, "On the Endomorphism Ring of a Semi-injective Module," Acta Math.Univ. Comenianae, Vol 71, 2002. pp 27-33.
- [17] S. Wongwai, "Almost Quasi-mininjective Modules," Chamjuri Journal of Mathematics, Vol 2, 2010, no. 1. pp 73-79.
- [18] S. Wongwai, "Small Principally Quasi-injective Modules," Int. J. Contemp. Math. Sciences, Vol 6, 2011, no. 11. pp 527-534.
- [19] S. Wongwai, "Quasi-small P-injective Modules," Journal of Science and Technology RMUTT, Vol 1, 2011, no. 1. pp 59-65.
- [20] Friedrich Kasch and Adolf Mader, "Rings, Modules and the Total," Birkhauser Verlag, Basel, Switzerland, 2004.
- [21] K. Amnuaykarn and S. Wongwai, "On Small Principall injective Rings," Proceeding of The 2nd International Science, Social Science, Engineering and Energy Conference 2010, December 15-16, 2010, Nakhonphanom, Thailand. pp 166.





Conference Proceeding Paper Title "On Small Principally injective Rings"

The 2nd International Science, Social Science, Engineering and Energy

Conference 2010 At Nakhonphanom River View Hotel December 15-16, 2010

Schedule of I-SEEC 2010

Date/Time	December 15, 2010	and the second	Salara Bassal	
8.00-12.00	Registration (Important notice: Please bring your registration receipt to the registration desk submit your full paper: 3 hard copies + a CD containing electronic file of you m verify your registration TODAY in order to avoid a long-waiting line due to a lar * Manuscript must be handed in before noon,	to verify your registra anuscript.) We encour ge number of conferen	tion status. After de rage EVERYONE te nce participants.	bing so, you m o registration o
9.30-10.00	Opening Ceremony MC: Dr.Ada Raimaturapong		3.5	E ale
	ROOM1	ROOM2	ROOM3	ROOM4
10.00-10.45	Plenary Lecture I : 'Nutrition in health and disease in dogs' Prof. Dr. Anton C.Beynen, Utrecht University, Netherlands.			
10.45-11.30	Plenary Lecture II : 'A PANDA Ring Resonator Design and Applications' Prof. Dr. Preecha Yupapin, Head of ARCP, KMITL, Thailand.	2		
11.30-12.15	Plenary Lecture III: 'Precision force measurement using the Levitation			

12.15-13.00	Contract of the strength of the		Lunch	
Time/Roo m	ROOM1	ROOM2	ROOM3	ROOM4
13.00-13.30	Invited Speaker: Functional Hybrid Optical Nanocomposites Based on Metal OxideOrganic- Nanostructure Materials Session Chair: Assoc. Prof. Dr. Wisanu Pecharapa	Invited Speaker: Research on Heat Transfer Enhancement in Refrigeration and Air Conditioning Session Chair: Part Dr. Somebal Wongwises	Invited Speaker: AS0014: Effects of Pen Types and Sizes on Production of DYL Cross-Bred Pigs Session Chair: Assist Prof Dr. Jakai Yaeram	Invited Speaker: SS0010: Rhythms of Village Life in Psychology of Rice Worship Ceremony Session Chair: Dr Kanopport Wonggarasin
13.30-13.45	OS0014: Ozone-Induced Optical Density Change of NiQ Thin Films and Their Applicability as Neutral Optical Density Filter R. Noonuruk N. Wongpisutpaisan P. Mukdacharoenchai W. Techitdheera W. Pecharapa	EE0012: A Biomarkers Study Trace Metal Elements in Paphia Undulate Shell for Assessing Pollution of Coastal Area N. Juncharoenwongsa W. Sinprom J. Kaewkhao A. Choeysuppaket P. Limsuwan K. Phachana	AS0002: Portable Electronic Nose and its Applications in <u>Vinevard</u> K. Teerakiat L. Panida	SS0001: Organizational Learning Sustaining the Competitive P.Chanthima P.Parnisara P.Darika
13.45-14.00	OS0015: All Optical Half AdderSubtractor using Dark- bright Soliton Conversion Control Conversion Control T. Sappasit P. Yupapin	EE0013: A Numerical Computation of Water Quality Measurement in a Uniform Channel Using a Finite Difference Method P. Nopparat D. Rujira	AS0003: Effect assessment of carcasses washing on the prevalence of Salmonella contamination at different stages of poultry slaughter Sana,M.j.	SS0003: The Influence of Organizational Support, Market Turbulence and Business Strategy on Market Orientation and New Product Activity P.Parnisara P.Chanthima P.Danika

Time/Room	ROOM1	ROOM2	ROOM3	ROOM4
14.00-14.15	OS0018: Determination of the thickness and optical constants of Z/O2 by spectroscopic ellipsometry and spectrophotometric method R. Yusoh M. Horprathum P. Eiamchai S. Chanyewadee K. Aiempanakit	EE0014: A Non-dimensional Form of Hydrodynamic Model with Variable coefficients in a Uniform Reservoir Using P. Nopparat S. Chunya	AS0004: Karvological and Randomly Amplified Polymorphic DNA-Polymerase Chain Reaction Studies in Barbodes spp. in Northeastern Thailand P.Keeravit T.Bungorn	S50005: Antecedents and Consequence of Organizational Commitment P. Surasit Y.Nikom P.Ampasri
14.15-14.30	OS0046: Design and Preparation of Synthetic Hydrogels Via Photopolymerisation for Biomedical Use as Wound Dressings W. Chinanat	EE0016: Characterization of PVA-Chitosan Nanofibers Prepared by Electrospinning K. Paipitak T. Pompra P. Mongkontalang W. Techitdheera W. Pecharapa	AS0005:Influence of Graded Level Protein of Cassava Root Meal Based Diets on Performance of Growing Pigs V.Ratchaneevan B.Wandee K.Nukon M.Charichai B.Suttipong	SS0008: Guideline on Community Radio Management for Social Givil Uttaradit Community Strength T.Rade C.Pannita Y.Sirikran
14.30-14.45	OS0006: A Novel Dynamic Optical Tweezers Array Generation Using Dark Soliton Control Within double AddDrop Multiplayer N. Pomsuwancharoen V. Thanyaphirak U. Punmeekeaw P.P. Yupapin	EE0018: Partial Discharge Monitoring System for Power Distribution Transformers as a Basis Risk Assessment Insulation Evaluation A.Promsak P.Winai P.Boonyang	AS0006: Effects of a hCG Administration on d 5 after a Timed Artificial Insemination on the Conception Rate of Postpartum Beef Cows in Small Holder Farmers S. Guntaprom J. Yaeram J. Jugsumit W. Pratoompol C. Amporn	SS0012: Perceptions of Professional Ethics in Thailand Conceptual Paper K.Thanit B.Sumintom
14.45-15.00		Coff	ee / Tea Break	The TRUE THE PARTY

- XXII -

4

Time/Room	ROOM1	ROOM2	ROOM3	ROOM4
15.00-15.30	Invited Speaker: SAFETY INSPECTION STRATEGY FOR EARTH MBANKMENT DAMS USING FULLY DISTRIBUTED SENSING by Prof. Dr. P.Y. Zhu	Invited Speaker: Future Global Visions of Engineering Education Invited Speaker by Assoc.Dr.Wisuit Sunthonkanokports	Invited Speaker: AS0024: Effect of fat type on feed intake rumen fermentation and nutrient digestibility in beef cattle	Invited Speaker: BU0008: Perceived Organizational Support on organizational commitment of Thai employees in rajabhat universities in the northerm group
	Session Chair: Assoc.Prof.Dr.Somsak Mitatha	Session Chair: Assoc.Dr.Wisuit Sunthonkanokpong	Session Chair: Assist. Prof. Assist.Prof. Dr. Chalermpon Yuangklang	Session Chair: Dr.Sumintom Baotham
15.30-15.45	Invited Speaker: Development of lead free radiation shielding glass experimental and theoretical approach by Dr. Jakrapong Kaewkhao	EE0023: Development of Electrical Transient Modeling and Simulation for Electric Distribution Systems B.Banyat S.Chitchai	AS0009: Biocontrol of Seed- Borne Pathogenic Fungi of Cabbage Seedling by Curcuma longa Extract <i>C.Piyanan</i>	BU0001: Audit Independence in Appearance in Thailand Conceptual Paper K.Payom B.Sumintom
15.45-16.00	EC0004: Theoretical calculation of optical absorption spectrum for Armchair graphene nanoribbon by first principle calculation <i>E. Ahmadi</i> <i>A. Asgari</i>	EE0029: Synthesis on the Nanoparticle of LaCoO3 Thermoelectric Material K.PASARA T.CHANCHANA S.TOSAWAT	AS0012: Frog Culture Development in Accordance with Sufficiency Economy Philosophy T.Thongyoon Y.Jaknt P.Wasan R.Wutii P.chaisongkrarm Poogingngem	BU0002: Antecedent and Consequence of Job Satisfaction and Organizational Commitment of Thai Employees in RMUTT Conceptual Paper B.Sumintom
		Contraction Contra	ABODT3: Effect of Giologically Elementari Dayto Josephili Elementari Biggionestation te Organisi Dany Court	

Time/Room	ROOM1	ROOM2	ROOM3	ROOM4
16.00.16.1 <mark>5</mark>	EC0007:The Wireless Electrical Load Automation Control System for Electrical Energy Saving N.Sarawut	NC0008: Image Recorder Server with IP Camera and Pocket PC N. Boonma A. Sangthong S. Mitatha and C. Vongchumyen	AS0013: Effect of Biologically Fermented Herbs Juices and Probiotics Supplementation in Organic Dairy Cows S.Nunthiya K.Jumrian P.Chaiya N.Nuttawut T.Nimlamai	BU0003: Effect of SCF from Operating Activities Format on Lenders' Decision Conceptual Paper Y.Arpa B.Sumintorn
16.15-16.30	EC0011: Small – Signal Model of Series – Parallel Resonant K.Chengchan	NC0009: SMS Information Display Board A. Tanadumrongpattana A. Suethakom S. Mitatha and C. Vongchumyen	AS0014: Effects of Pen Types and Sizes on Production of DYL Cross-Bred Pigs Y.Jakrit P.Wasun S.Jirasak D.Sarawut	BU0005: Innovation Capability, Market Orientation Constructs and Business Performance in Thailand Conceptual Paper W.Duangrudee B.Sawitee
16.30-16.45	MT0002: On Small Principally Injective Rings K. Amnuaykarn S. Wongwai	NC0010: Wireless Traffic Light Controller K. Thatsanavipas N. Ponganunchoke S. Mitatha and C. Vongchumyen	AS0015: Karyological and RAPD-PCR Studies in Barbodes spp P.Keeravit T.Bungorn	BUG006: National Culture on Accounting Values Gray's Constructs Conceptual Paper Phase 1 T.Wimolyai B.Sumintorn K.Lertluk
16.30-17.00		Poster	Board Part I	
17.00-18.00	and the second se	Presen	tation Poster	
18 30-22 00	Wel	come Party (swimming pool-sid	te beer garden ground floor kk	ong river)

December 16	, 2010		16 J	a second second second second
Time/Room	ROOM1	ROOM2	ROOM3	ROOM4
9.00-10.00	Poster Sesion Part2 / Free Time Presentation Poster	Plenary Lecture IV: Prof. Dr. Koichi KAKU, AFFFRC, Japan. 'Global warming and climate change of Asian Countries including Japanese domestic GHG emission from the standpoint of Clean Developing Mechanism (CDM) for Green House Gas (GHG) reduction in the Field of Agriculture'	Plenary Lecture V: Dr. Sudarath Sakhunkhu, RMUTI, Thailand, 'Yield Trial of 6 Varieties of Azukibean in Sakhon-Nakhon'	
10.00-10.30	Poster Sesion Part2 / Free Time Presentation Poster	Invited Speaker: EC0008: Harmonic Suppression Improvement of Microstrip Open Loop Ring Resonator Bandpass Filter Session Chair: Dr.Ravee Bhomhoundsri	Invited Speaker:AS0018: Effect of Location for raised on Growth Performance and Carcass Quality of Kadon Pig Session Chair: Assita Prof. Dr. Kraisit Vasupen	Invited Speaker:BU0004: Effects of Organizational Alliance Success on Performance of Hotel Industry in Thailand Session Chair: Dr.Chanthima Phromket
10.30-10.45	2	Coffee	/ Tea Break	
10.45-11.00	Poster Sesion Part3 / Free Time Presentation Poster	EE0031: The Effect of Jumbo Compact Fluorescent Lamp Energy Saving Lamp on the Electrical Energy Saving and Harmonics Noise U.Chutipon	AS0019: Interactive effects of the feeding of cassava leaves and calcium level on feed intake and macronutrient digestion in beef cattle J. Khotsakdee N. Opatawong K. Vasupen S. Bureenok S. Wongsuthavas P. Panvakaew C. Yuangkiang	BU0007: National Culture on Accounting Values Grav s Conservatism and Secrecy Hypothesis Conceptual Paper Phase 2 H.Wilaiporn B.Sumintorn K.Nuchnapar

Time/Room	ROOM1	ROOM2	ROOM3	ROOM4
11.00-11.15		EE0032:Side Effect of T5 Ballast on the Conducted Electromagnetic Interference or EMI and Harmonics Noise U.Chutipon	AS0021:In vitro gas production measurements to evaluate different fat sources based on urea-treated rice straw as main roughage source J. Khotsakdee P. Hunghuan K. Vasupen S. Bureenok S. Wongsuthavas P. Panyakaew C. Yuangklang	BU0009:Professional Ethic on Audit or and Public Perception in Thailan d (1) K.Thanit B.Sumintorn
10.00-19-45		EE0036:The Effect of Power	AS0024:Effect of fat type on	
11.15-11.30		Supply of LED Lamp K.Patiphan K.Chengchan U.Chutipon	feed intake rumen fermentation and nutrient digestibility in beef cattle K. Kongweha K. Vasupen P. Paengkoum S. Bureenok S. Wongsuthavas C. Yuangklang	
11.30-11.45	•		AS0010:Growth rate, heat tolerance, dressing percentage and rib eye area of Brahman and Angus crossbred steers under conventional fattening fed rice straw based diet in Thailand C. Promkot P. Pornanake	
12.00.12.00			Finany Lecture V.	
12.00-13.00			Lunch	

	Contents	Page.
Program (Dverview	XVI
Schedule o	f I-SEEC 2010	xx
Agricultu	ral science	
AS0001	Method Development for Cadmium, Lead and Zinc Determination in	1
	Soils and Vegetables Collected from Mae Sot, Tak Province	
AS0002	Portable Electronic Nose and its Applications in Vineyard	2
AS0003	Effect assessment of carcasses washing on the prevalence of	4
	Salmonella contamination at different stages of poultry slaughter	
AS0004	Karyological and Randomly Amplified Polymorphic DNA-Polymerase	5
	Chain Reaction Studies in Barbodes spp. in Northeastern Thailand	
AS0005	Influence of Graded Level Protein of Cassava Root Meal Based Diets	6
	on Performance of Growing Pigs	
AS0006	Effects of a hCG Administration on d 5 after a Timed Artificial	7
	Insemination on the Conception Rate of Postpartum Beef Cows in	
	Small Holder Farmers	
AS0007	The Development of Pelleting Machine for Fish Feed Containing	8 2
	Cassava as a Main Source	
AS0008	Technology Transfer of Water Application from Hybrid Walking	9
	Catfish (Clarias macrocepharus x Clarias gariepinus) Culture for	
	Aquaponics Production	
AS0009	Biocontrol of Seed-Borne Pathogenic Fungi of Cabbage Seedling by	10
	Curcuma longa Extract	
AS0010	Growth rate, heat tolerance, dressing percentage and rib eye area of	11
	Brahman and Angus crossbred steers under conventional fattening fed	
	rice straw based diet in Thailand	
AS0011	Effect of Inorganic Soil Amendment (Zeolite) on Yields of RRIT 251	12
	and RRIM 600 Para Rubber	
AS0012	Frog Culture Development in Accordance with Sufficiency Economy	13
	Philosophy	

- XXX -

AS0013	Effect of Biologically Fermented Herbs Juices and Probiotics	14
	Supplementation in Organic Dairy Cows	
AS0014	Effects of Pen Types and Sizes on Production of DYL Cross-Bred Pigs	15
AS0015	Karyological and RAPD-PCR Studies in Barbodes spp	16
AS0016	Effects of feeding the graded level protein of cassava root meal diets	17
	with the four first-limiting amino acids adjustments on performance	
	and some carcass traits of finishing pigs	
AS0017	Utilization of Ectomycorrhiza for Cash Crops Plantation	18
AS0018	Effect of Location for raised on Growth Performance and Carcass	19
	Quality of Kadon Pig	
AS0019	Interactive effects of the feeding of cassava leaves and calcium level on	20
	feed intake and macronutrient digestion in beef cattle	800033
AS0020	Effect of addition fermented juice of epiphytic lactic acid bacteria	21
	(FJLB)to fermented total mixed ration (FTMR) on the in vitro dry	
	matter digestibility	
AS0021	In vitro gas production measurements to evaluate different fat sources	22
	based on urea-treated rice straw as main roughage source	
AS0022	In vitro gas production measurements to evaluate different fat sources	23
	based on rice straw as main roughage source	
AS0023	Effects of Dietary Acid Supplementation and Difference Environmental	24
	Temperature on Growth Performance of Broiler Chickens	
AS0024	Effect of fat type on feed intake, rumen fermentation and nutrient	25
*	digestibility in beef cattle	
AS0025	Sponge Cake from Composite Cassava-Wheat Flour Fortified with	26
	Mulberry Leaf	
AS0026	The Research and Development Project of Frog Culture Method with	27
	Sufficiency Economy Philosophy	
	Study on Electronic Structure of a Last There Sectors Minerial for	
23	Partial Discharge Mandoring System for Power Distribution	

- XXXI -

Sher By an	There of the subscription	
EE0001	Remote Terminal Air-Conditioner Unit for Building Energy-Saving	28
EE0002	Desulfurization of Waste Tire Pyrolysis Oil via Photo-oxidation	29
	Catalyzed by Titanium Dioxide	
EE0003	A Biomarkers Study: Trace Metal Elements in Paphia Undulate Shell	31
	for Assessing Pollution of Coastal Area	
EE0004	The effect of calcinations of diatomite to adsorption of chromate	32
EE0005	Calcinations effect of diatomite to chromate adsorption	33
EE0006	Investigation of Biomass Fly Ash in Thailand for Recycle to Glass	34
EE0007	An optimized PV Monitoring System for the bus shelter	35
EE0008	A new design double solar energy tower conception for	36
	Combination/hybrid system	1050084
EE0009	Potential of using a Solar-Electricity Hybrid System in North-East of	37
	Thailand	
EE0010	Fabrication of Alkali Borosilicate Glass using Fly Ash from Industrial	38
	Waste yonu Winds gram Winte yon botasot-sem no board	
EE0012	A Biomarkers Study: Trace Metal Elements in Paphia Undulate Shell	39
	for Assessing Pollution of Coastal Area	*
EE0013	A Numerical Computation of Water Quality Measurement in a Uniform	40
	Channel Using a Finite Difference Method	
EE0014	A Non-dimensional Form of Hydrodynamic Model with Variable	41
	coefficients in a Uniform Reservoir Using Lax-Wendroff Method	
EE0015	Design and Construction of 2.45 GHz Microwave Plasma Source	· 42
	at atmospheric pressure	
EE0016	Characterization of PVA-Chitosan Nanofibers Prepared	43
	by Electrospinning	
EE0017	Study on Electronic Structure of In2Te3 Thermoelectric Material for	44
	Alternative Energy	
EE0018	Partial Discharge Monitoring System for Power Distribution	45
	Transformers as a Basis Risk Assessment Insulation Evaluation	
EE0019	Model and Experiment Analysis of 1.2 kW PEMFC Electrification	46

- XXXII -

EE0020	Analysis of energy consumption and behavior of Television in the brue	47
	resident houses in Thailand	
EE0021	A Practical Method for Quickly PV sizing	48
EE0022	A Design of Biogas Fermentation Tank from Banana Shell	49
EE0023	Development of Electrical Transient Modeling and Simulation for	50
	Electric Distribution Systems	
EE0024	Plant pot production from the leaves of sugarcane	51
EE0025	Drying of Homtong parboiled rice by hot from boiling stove	52
EE0026	Thermoelectric Device	53
EE0027	Design and Construction of Generators and Refrigeration from	54
	Thermoelectric cells	
EE0028	Design and constriction of A Mobile PV Hybrid System Prototype for	55
	isolated electrification	
EE0029	Synthesis on the Nanoparticle of LaCoO3 Thermoelectric Material	56
EE0030	Design and Construction Chamber and Mechanical for Bulk	57
	Thermoelectric Property Measurements	
EE0031	The Effect of Jumbo Compact Fluorescent Lamp (Energy Saving	58
	Lamp) on the Electrical Energy Saving and Harmonics Noise	
EE0032	Side Effect of T5 Ballast on the Conducted Electromagnetic	59
	Interference or EMI and Harmonics Noise	
EE0033	Study on Thermoelectric Properties of ZnO Nanoparticles	60
EE0034	Analyzing of Thermoelectric Refrigerator Performance	61
EE0035	The Wireless Electrical Load Automation Control System for	62
	Electrical Energy Saving	
EE0036	The Effect of Power Supply of LED Lamp for External lighting	63
EE0037	Characterization on Nano and Micro Crystals of ZnO	64
EE0038	Future Global Visions of Engineering Education	65

- XXXIII -

Electroni	c and Communication	
EC0001	Efficiency Enhancement of Dual-Band Bandpass Filter with Inductive	66
85	Compensated Coupled Line and the telephone and body the telephone A	
EC0002	A COMPACT FOUR-POLE CROSS COUPLE SQUARE OPEN	67
	LOOP WITH ASYMMTRIC FEED	
EC0003	A Novel configurations of op-amp oscillator using only unity-gain	68
	voltage follower.	
EC0004	Theoretical calculation of optical absorption spectrum for Armchair grapheme nanoribbon	69
EC0005	FM Radio Broadcasting Transmitting using Triple Frequency on Single Radio Frequency Amplifier Module and Single Antenna System	70
EC0006	The RFID Application for Electrical Energy Saving in Office	71
EC0007	The control of electrical energy consumption using wireless automated	- 72
FC0008	Harmonic Suppression Improvement of Micro strip Open Loop Ring	73
Levou	Resonator Band pass Filter	
EC0009	Design of compact micro strip stepped-impedance resonator band pass	74
EC0010	RGB color correlation index for image Retrieval	75
EC0011	Small - Signal Model of Series - Parallel Resonant DC-DC Converter	76
	with Capacitive Output Filter	
	Analyzing of being define a state for the second of the	
Network '	Technologies and Computation Intelligence	
NC0001	USB Security Camera Software for Linux	• 77
NC0002	Design of information vehicle for tracking vehicle missing which based	78
	upon GPRS technology	
NC0003	Design of information location for coordinate specifying which based upon GPRS technology	79
NC0004	Integrated inventory-routing problem in one warehouse and multi- retailer distribution system	80

- XXXIV -

NC0006	Design of information CCTV in snapshot for a saving to backup site	81
	which upon GPRS technology	
NC0007	A new design of information in transport layer for protocol by	83
	advantage of TCP and UDP method	
NC0008	Image Recorder Server with IP Camera and Pocket PC	84 ·
NC0009	SMS Information Display Board	85
NC0010	Wireless Traffic Light Controller	86
	Gaundan pulse generated und maltipleadel by using a Add/Drop filter	
Optical Sc	sience and Technology	
OS0001	An Optical Electronic Nose System Based on Organic Sensor for	87
	Beverages Analysis	
OS0002	Optical and Structural Investigation of Bismuth Borate Glasses Doped	89
	With Dy3+	
OS0003	Effects of Side Chain Length and Head Group Structure on Color	91
	Switching of Polydiacetylene Vesicles	
OS0004	Controlling the Color Switching of Polydiacetylene Vesicles by	92
	Adjusting Diacetylene Monomer Structure	
OS0005	Colorimetric UV radiation Sensors using Organic Dye Thin Films	94
OS0006	A Novel Dynamic Optical Tweezers Array Generation Using Dark	95
	Soliton Control Within double Add/Drop Multiplayer	
OS0007	A novel design of the nonlinear nanoring resonator systems and	96
	potential applications	
OS0008	A novel design of the nonlinear microring resonator systems for	97
	smallest cutting cancer applications	
OS0009	A novel design of the nonlinear microring resonator systems for THz	98
	communication system applications	
OS0010	A new design DWDM convert broadband THz communication by the	99
	nonlinear microring resonator systems	
OS0011	A new design double solar energy conception for combination system	100
OS0012	Quantum Memory using the Multi-single-photons Storage within a	101
	Nano-waveguide System for Security Camera Use	

- XXXV -

OS0013	Preparation and Properties of Bi2O3-B2O3-Nd2O3 Glass System	102
OS0014	Ozone-Induced Optical Density Change of NiO Thin Films and Their	103
	Applicability as Neutral Optical Density Filter	
OS0015	All Optical Half Adder/Subtractor using Dark-bright Soliton	104
	Conversion Control	
OS0016	Information System Development, OTOP In Kalasin Province	105
	Organization Website Structure	
OS0017	Gaussian pulse generated and multiplexed by using a Add/Drop filter	106
	within a waveguide system	
OS0018	Determination of the thickness and optical constants of ZrO2 by	107
	spectroscopic ellipsometry and spectrophotometric method	
OS0019	Orthogonal Photon States Manipulation using Dark-Bright Soliton	108
.68	Conversion Control	080002
OS0020	A new Refinement of used lubricant as Renewable fuel of diesel fuel	109
OS0021	Quantum Gates from Ultra-Short Pulses in Fiber Optics	110
OS0022	Sol-gel based deposition of TixV1-xO films for thermally controlled	111
	optical switching applications	
OS0023	Novel Multi Optical Trapping Tool Generation within Add/Drop Filter	112
	System Controlled by Light	CINCIPAL D
OS0024	Polarization state control by using rotating quarter wave plate for the	113
	measurement using by light	
OS0025	The measurement of ellipsometric parameter of various liquid using a	114
	polarization state control technique	
OS0026	Drug Trapping and Delivery Using a PANDA Ring Resonator	.115
OS0027	Numerical Simulation Optical Buffer of Microring Resonator 1.5µm	116
	Radius Array	
OS0028	Photon Switching using a Nonlinear PANDA Ring Resonator	117
OS0029	Dynamic Optical Tweezers Generation using a PANDA Ring	118
	Resonator	
OS0030	Multi Light Sources Enhancement using Double PANDA Ring	119
	Resonators	
OS0031	Data Security Transmission via a Noisy Channel	120

- XXXVI -

OS0032	Quantum Synchronization for Multi Variable Packet Switching	121
	Security annota schinger 2 an grintering guines I freehestung?	
OS0033	Absorption and Coloration of MnO2 doped in Soda-Lime-Silicate and	122
	Soda-lime-borate Glasses	
OS0034	Functional Hybrid Optical Nanocomposites Based on the Internet State	123
	Metal Oxide /Organic-Nanostructure Materials	
OS0036	Development of Two Pellet Die Organic Fertilizer Compression	124
	Machine and Island to other The main and all trained and and and a more the	
OS0037	Microwave Dielectric Measurement of liquids by using Waveguide	125
	Plunger Technique	
OS0038	Phosphorus Value Determination in Mao-Wine by Spectrophotometry	126
OS0039	The Variation of Energy Gap of NiO under Pressure Changed by First	127
	Principle Calculation	
OS0040	Energy Gap Transition of Paramagnetic NiO under Pressure	128
OS0043	Effect of interface recombination on the performance of SWCNT\GaAs	129
	heterojunction solar cell	
OS0044	An Analytical Model for Detectivity Prediction of Uncooled Bolometer	130
	Considering all Thermal Phenomena Effects	850002
OS0045	Development of lead free radiation shielding glass: experimental and	131
	theoretical approach	
OS0046	Design and Preparation of Synthetic Hydrogels Via hotopolymerisation	132
	for Biomedical Use as Wound Dressings	
OS0047	All-Optical Data Comparison with dark-bright soliton conversion	133
	control Design of the minimum and the	
	Study of Khida and the and the and	
Network	Technologies and Computation Intelligence	
PE0001	Fabrication and Characterization of B(Pb)SCCO Superconducting	135
	Whisker Josephson Junction	
154		

Distaint, Udon Thani Provident

- XXXVII -

Social Sci	Ocontum Synchronization for Fiditi Variable Packet Switching 9209	
SS0001	Organizational Learning: Sustaining the Competitive Advantage	136
	Gained Through Innovation Leverage	
SS0002	Effects of personal Behaviors on SMEs	137
	Accountant' Professional Ethics in Kalasin province	
SS0003	The Influence of Organizational Support, Market Turbulence and	139
	Business Strategy on Market Orientation and New Product Activity	
SS0004	Factors Affecting Ethical Behaviors of Faculty of Social Technology	140
	Students, RMUTI Kalasin Campus	
SS0005	Antecedents and Consequence of Organizational Commitment: The	141
	Role of External locus of control	REAMED
SS0006	Motivation in Bachelor Degree Studying in Faculty of Social	142
	Technology for Higher Vocational course Students in Kalasin	
	Province meaning the second of	
SS0007	The Development of Teaching and learning Activity on Limit and	144
	Differentiation Algebraic Function for Undergraduate Students by	
	Computer Assisted Instruction lesson	
SS0008	Guideline on Community Radio Management for Social Civil Uttaradit	146
	Community Strength Concept Paper	DSIN IS
SS0009	Characterization of Nickel Oxide Thin Films Prepared Spin-Coating	147
	Process	
SS0010	Ancestor worship the sacred spirit	148
SS0011	The Structural Equation Theory of Planned Behavior and Past Behavior	150
	on Altruism Acts and Organizational Commitment Model: The Case	
	Study of Khon Kaen Brewery Co., Ltd.	
SS0012	Perceptions of Professional Ethics in Thailand: Conceptual Paper	151
SS0013	Management System Models to Support Decision-making for Micro	152
	and Small Business of Rural Enterprise in Thailand	
SS0014	Skills for the IT services industry in latecomer countries	153
SS0015	Academic-Service Project - Management for the Community Ban	154
	Dong Kham Occupational Group, Phon-Ngam Sub-district, Nong Harn	
	District, Udon Thani Province	

- XXXVIII -

SS0016	The Effect of Market Orientation and Marketing Strategy Adaptation	155
	on Market Performance: The Role of Internal and External	
	Contingency as a Moderator	
Business		
BU0001	Audit Independence in Appearance in Thailand: Conceptual Paper	156
BU0002	Antecedent and Consequence of Job Satisfaction and Organizational	157
	Commitment of Thai Employees in RMUTT: Conceptual Paper	
BU0003	Effect of SCF from Operating Activities Format on Lenders' Decision: Conceptual Paper	158
BU0004	Effects of Organizational Alliance Success on Performance of Hotel	159
	Industry in Thailand	
BU0005	Innovation Capability, Market Orientation Constructs and Business	160
	Performance in Thailand: Conceptual Paper	
BU0006	National Culture on Accounting Values: Gray's Constructs (Conceptual	161
	Paper: Phase 1)	
BU0007	National Culture on Accounting Values: Gray's Conservatism and	162
	Secrecy Hypothesis (Conceptual Paper: Phase 2)	ite .
BU0008	Perceived Organizational Support and Organizational Commitment:	163
	Conceptual Paper	
BU0009	Perceptions of Professional Ethics in Thailand: Conceptual Paper	164
ni batupean Others		
others	a Barrier Barrier Barrier Barrier Barrier Barrier	offs and ee
MT0001	Numerical Methods and Programming for Solving Nonlinear Equations	165
MT0002	On Small Principally Injective Rings	166

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- XXXIX -

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On Small Principally Injective Rings

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Abstract

Let R be a ring. A right R-module M is called *small principally injective* (briefly, SP-injective) if, every R-homomorphism from a small and principal right ideal aR to M can be extended to an R-homomorphism from R to M [11]. A ring R is called right SP-injective if, R_R is SP-injective. In this paper, we give some characterizations and properties of small principally injective modules and small principally injective rings.

Keywords: Professional Ethics.

1. Introduction

Throughout this paper, R will be an associative ring with identity and all modules are unitary right R-modules. For right R-modules M and N, $\operatorname{Hom}_R(M, N)$ denotes the set of all R-homomorphisms from M to N and $S = End_R(M)$ denotes the endomorphism ring of M. By notations, $N \subset^{\oplus} M$, $N \subset^{e} M$, and $N \ll M$ we mean that N is a direct summand, an essential submodule and a superfluous submodule of M, respectively. We denote the socle and the Jacobson radical of M by Soc(M) and J(M), respectively.

Let R be a ring. A right R-module M is called *principally injective* (or P-injective), if every R-homomorphism from a principal right ideal of R to M can be extended to an R-

homomorphism from R to M. Equivalently, $l_{M'R}(a) = Ma$ for all $a \in R$, where l and r are the left and right annihilators, respectively. This notion was introduced by Camillo [2] for commutative rings. In [7], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. They also continued to study rings with some other kind of injectivity, namely, mininjective rings [8]. A ring R is called *right mininjective* if every isomorphism between simple right ideals is given by left multiplication by an element of R. Equivalently, if kR is simple, $k \in R$, lr(k) = Rk. In [11], L.V. Thuyet, and T.C. Quynh, introduced a small principally module. A right R-module M is called *small principally injective (or SP-injective)* if, every R-homomorphism from a small and principal right ideal aR to M can be extended to an R-homomorphism from R to M. In this paper we also consider small principally injective modules and rings.

Following [1], a submodule K of a right R-module M is superfluous (or small) in M, abbreviated $K \ll M$, in case for every submodule L of M, K + L = M implies L = M. It is clear that $aR \ll R$ if and only if $a \in J(R)$.

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2. SP-injective Modules

Definition 2.1. [11] Let R be a ring. A right R- module M is called *small principally*

injective (briefly, SP-injective) if, every R-homomorphism from a small and principal right ideal aR to M can be extended to an R-homomorphism from R to M.

Lemma 2.2. Let M be a right R-module. Then M is SP-injective if and only if each $a \in J(R)$, $l_M r_R(a) = Ma$.

Proof. Clearly, $Ma \subset l_M r_R(a)$. Let $x \in l_M r_R(a)$. Define $\varphi : aR \to xR$ by $\varphi(ar) = xr$, for every $r \in R$. Since $r_R(a) \subset \varphi_R(x)$, φ is well-defined so it is clear that φ is an R-homomorphism. Since M is SP-injective, there exists an R-homomorphism $\widehat{\varphi} : R \to M$ such that $\widehat{\varphi}\iota_2 = \iota_1\varphi$, where $\iota_1 : xR \to M$ and $\iota_2 : aR \to R$ are the inclusion maps. Then $x = \varphi(a) = \widehat{\varphi}(1)a \in Ma$.

Conversely, let $a \in J(R)$, and let $\varphi : aR \to M$ be an R-homomorphism. Then $\varphi(a) \in l_M r_R(a)$, so by assumption, we have $\varphi(a) = xa$ for some $x \in M$. Define $\hat{\varphi} : R \to M$ by $\hat{\varphi}(r) = xr$ every $r \in R$. It is clear that $\hat{\varphi}$ is an R-homomorphism and is an extension of φ .

Example 2.3. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field, and $M_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then M is SP-injective.

Proof. It is clear that only $A = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is the nonzero small principal right ideal of R. Let $0 \neq a \in A$. Then $r_R(a) = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ so $l_{MTR}(a) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. It is obvious that $Ma = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Then by Lemma 2.2, M is SP-injective.

Proposition 2.4. Let $\{M_i : i \in I\}$ be a family of right R-modules. Then the direct product $\prod_{i \in I} M_i$ is SP-injective if and only if each M_i is SP-injective.

Proof. Let π_i and φ_i , for each $i \in I$, be the ith projection map and the ith injection map, respectively. We now let $i \in I$, $a \in J(R)$, and let $\varphi : aR \to M_i$ be an R-homomorphism. Then by assumption, there exists an R-homomorphism $\widehat{\varphi} : R \to M_i$ such that $\widehat{\varphi}_i = \varphi_i \varphi$ where $i : aR \to R$ is the inclusion map. Thus $\varphi = \pi_i \widehat{\varphi}_i$. Conversely, let $a \in J(R)$ and $\varphi : aR \to \prod_{i \in I} M_i$ be an R-homomorphism. Then for each $i \in I$, there exists an R-homomorphism $\alpha_i : R \to M_i$ such that $\alpha_i \iota = \pi_i \varphi$ where $\iota : aR \to R$ is the inclusion map. Hence we obtain (product) $\widehat{\varphi} : R \to \prod_{i \in I} M_i$ with $\pi_i \widehat{\varphi} = \alpha_i$ and $\pi_i \widehat{\varphi} \iota = \alpha_i \iota$ which implies $\widehat{\varphi}_i = \varphi$.

Lemma 2.5. Let M_i $(1 \le i \le n)$ be SP-injective modules. Then $\oplus_{i=1}^n M_i$ is SP-injective.

Proof. It is enough to prove the result for n = 2. Let $a \in J(R)$ and $\varphi : aR \to M_1 \bigoplus M_2$ be an *R*-homomorphism. Since M_1 and M_2 are *SP*-injective, there exists an *R*-homomorphisms $\varphi_1 : R \to M_1$ and $\varphi_2 : R \to M_2$ such that $\varphi_1 \iota = \pi_1 \varphi$ and $\varphi_2 \iota = \pi_2 \varphi$ where π_1 and π_2 are the projection maps from $N_1 \bigoplus N_2$ to N_1 and N_2 , respectively, and $\iota : aR \to R$ is the inclusion map. Set $\hat{\varphi} = \iota_1 \varphi_1 + \iota_2 \varphi_2 : R \to M_1 \bigoplus M_2$. Thus it is clear that $\hat{\varphi}$ extends φ . \Box

Lemma 2.6. Any direct summand of an SP-injective module is again SP-injective.

Proof. By definition.

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3. SP-injective Rings

If R_R is an SP-injective module, then we call R is a right SP-injective ring. In this section, we give some properties and characterizations of SP-injective rings.

The following lemma follows from Lemma 2.2.

Lemma 3.1. Let R be a ring. Then R is right SP-injective if and only if each $a \in J(R)$, lr(a) = Ra.

Theorem 3.2. Let R be a right SP-injective ring. Then

- (1) R is right mininjective.
- (2) lr(Ra) = Ra, for any a ∈ J(R),

(3) If aR ⊕ bR and Ra ⊕ Rb are both direct, a, b ∈ J(R), then l(a) + l(b) = R.

Proof. (1) Since every simple right ideal of R is either nilpotent or a direct summand of R [4, (10.22) Brauer's Lemma], each right SP-injective ring is right mininjective ring.

(2) Let $x \in lr(Ra)$. Define $\varphi : aR \to xR$ by $\varphi(ar) = xr$ for every $r \in R$. Since ar = 0 implies xr = 0, φ is well-defined. It is clear that φ is an R-homomorphism. Since R is right SP-injective, there exists an extension $\hat{\varphi} : R \to R$ of φ . Hence $x = \varphi(a) = \hat{\varphi}(1)a \in Ra$. This shows that $lr(Ra) \subset Ra$. The inclusion $Ra \subset lr(Ra)$ is always holds.

(3) Define $\varphi : (a+b)R \to R$ by $\varphi(a+b)r = br$ for every $r \in R$. If (a+b)r = 0, then $ar = br \in aR \cap bR = 0$ so br = 0. This shows that φ is well-defined. It is clear that φ is an R-homomorphism. Then there exists an extension $\widehat{\varphi} : R \to R$ of φ . Hence $\widehat{\varphi}(1)(a+b) = \varphi(a+b) = b$ so $\widehat{\varphi}(1)a = (1-\widehat{\varphi}(1))b \in Ra \cap Rb = 0$. Then $\widehat{\varphi}(1) \in l(a)$ and $(1-\widehat{\varphi}(1)) \in l(b)$. Hence $1 = \widehat{\varphi}(1) + (1-\widehat{\varphi}(1)) \in l(a) + l(b)$. It follows that l(a) + l(b) = R.

Proposition 3.3. If R is right SP-injective, so is eRe for all $e^2 = e \in R$ satisfying ReR = R.

Theorem 3.4. Let R be right SP-injective, $a \in R$ and $b \in J(R)$.

- (1) If bR embeds in aR, then Rb is an image of Ra.
- (2) If aR is an image of bR, then Ra embeds in Rb.
- If bR ≃ aR, then Ra ≃ Rb.

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Proof. (1) Let $f: bR \to aR$ be an R-monomorphism. Let $\iota_1: bR \to R$ and $\iota_2: aR \to R$ be the inclusion maps. Since R is right SP-injective, there exists an R-homomorphism $\widehat{f}: R \to R$ such that $\iota_2 f = \widehat{f}\iota_1$. Let $\sigma: Ra \to Rb$ defined by $\sigma(sa) = s\widehat{f}(b)$ for every $s \in R$. If sa = 0, then $\sigma(sa) = s\widehat{f}(b) = sf(b) \in s(aR) = (sa)R = 0$. This shows that σ is well-defined. It is clear that σ is an R-homomorphism. Note that $\widehat{f}(b)R \ll R$ by [1, Lemma 5.18]. For any $s \in \overline{R}$, $\widehat{f}(b)s = 0$ implies $\widehat{f}(bs) = 0$ so bs = 0 because f is monic. Consequently, $r(\widehat{f}(b)) \subset r(b)$ and hence $\overline{tr}(b) \subset tr(\widehat{f}(b))$. Then by Lemma 3.1, $Rb \subset R\widehat{f}(b)$. Thus $b \in R\widehat{f}(b)$ and so $b = s\widehat{f}(b) = \sigma(sa)$.

(2) By the same notations as in (1), let $f : bR \to aR$ be an R-epimorphism. Since R be right SP-injective, f can be extended to $\hat{f} : R \to R$ such that $\iota_2 f = \hat{f}\iota_1$. Write $a = f(bx) = \hat{f}(bx), x \in R$. Define $\sigma : Ra \to Rb$ by $\sigma(sa) = s\hat{f}(bx)$ for every $s \in R$. It is clear that σ is an R-homomorphism. If $sa \notin Ker(\sigma)$, then $0 = \sigma(sa) = s\hat{f}(bx) = sf(bx) = sa$. Hence σ is an R-monomorphism.

(3) Follows from (1) and (2).

Following [1], a ring is R semiprimitive in case J(R) = 0.

Proposition 3.5. The following conditions are equivalent for a ring R:

- (1) R is semiprimitive.
- (2) Every right R-module is SP-injective.
- (3) Every principal right ideal is SP-injective.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ is clear.

(3) \Rightarrow (1) Suppose $J \neq 0$. Then there exists a nonzero element $a \in J(R)$. Then by assumption, the inclusion map from aR to R is split. Then aR is a direct summand of R so aR = 0 which is a contradiction.

References

- F. W. Anderson and K. R. Fuller, Rings and Categories of Modules," Graduate Texts in Math.No.13 , Springer-verlag, New York, 1992.
- [2] V. Camillo, Commutative rings whose principal ideals are annihilators, Portugal Math., 46 (1989) 33-37.
- [3] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, Extending Modules, Pitman, London, 1994.
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