QUASI-SMALL PRINCIPALLY-INJECTIVE MODULES

PASSAKORN YORDSORN

A THESIS SUBMITTED IN PARTIAL FULLFILLMENT OF THE

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Thesis Title	Quasi-Small Principally-Injective Modules
Name - Surname	Mr. Passakorn Yordsorn
Program	Mathematics
Thesis Advisor	Assistant Professor Sarun Wongwai, Ph.D.
Academic Year	2012

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Date...10...Month…March…Years…2013…

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ABSTRACT

The purposes of this thesis are to (1) study properties and characterizations of quasi-small principally-injective modules, (2) study properties and characterizations of endomorphism rings of quasi-small principally-injective modules, (3) extend the concepts of quasi-principally injective modules, and (4) find some relations between quasi-principally injective modules, quasi-small principally-injective modules and projective modules.

Let R be a ring. A right R-module M is called *principally injective* if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. A right R-module N is called *M-principally injective* if every R-homomorphism from an M-cyclic submodule of M to N can be extended to an R-homomorphism from M to N. A right R-module M is called *quasi-principally injective* if it is M-principally injective. The notion of quasi-principally injective modules is extended to be quasi-small principally-injective modules. A right R-module N is called *M-small principally-injective* if every R-homomorphism from an M-cyclic small submodule of M to N can be extended to an R-homomorphism from M to N. A right R-module M is called *quasi-small principally-injective* if it is M-small principally-injective.

The results are as follows. (1) The following conditions are equivalent for a projective module M : (a) every M-cyclic small submodule of M is projective; (b) every factor module of an M-small principally-injective module is M-small principally-injective; (c) every factor module of an injective R-module is M-small principally-injective. (2) Let M be a right R-module and $S = End_R(M)$. Then the following conditions are equivalent: (a) M is quasi-small principally-injective; (b) $l_S(Ker(s)) = Ss$ for all $s \in S$ with $s(M) \ll M$; (c) $Ker(s) \subset Ker(t)$, where $s, t \in S$ with $s(M) \ll M$, implies $St \subset Ss$; (d) $l_S(Ker(s) \cap Im(t)) = l_S(Im(t)) + Ss$ for all $s, t \in S$ with $s(M) \ll M$. (3) Let M be a principal module which is a self generator. If M is quasi-small principally-injective, then: (a) if $sS \oplus tS$

and $Ss \oplus St$ are both direct, $s, t \in J(S)$, then $l_M(s) + l_M(t) = S$; (b) $l_S r_M(Ss) = Ss$ for any $s \in J(S)$. (4) Let M be a quasi-small principally-injective module, $s, t \in S$ and $s(M)_d \ll M$: (a) if $s(M)$ embeds in $t(M)$, then Ss is an image of St; (b) if $t(M)$ is an image of $s(M)$, then St embeds in Ss; (c) if $s(M) \cong t(M)$, then $Ss \cong St$.

Keywords: quasi principally-injective modules, quasi-small principally-injective modules, endomorphism rings

หัวข้อวิทยานิพนธ์	มอดูลแบบควอซี-สมอลพรินซิเพ็ลลิ-อินเจคทีฟ
ชื่อ - นามสกุล	นายภาสกรณ์ ยอดสอน
สาขาวิชา	คณิตศาสตร์
อาจารย์ที่ปรึกษา	ผู้ช่วยศาสตราจารย์ ศรัณย์ ว่องไว, วท.ด.
ปีการศึกษา	2555

บทคัดย่อ

วิทยานิพนธ์นี้มีวัตถุประสงค์เพื่อ (1) ศึกษาสมบัติและลักษณะเฉพาะของ ควอซี-สมอลพรินซิเพ็ลลิ-อินเจคทีฟมอดูล (2) ศึกษาสมบัติและลักษณะเฉพาะของริงอันตรสัณฐานของ ควอซี-สมอลพรินซิเพ็ลลิ-อินเจคทีฟมอดูล (3) ขยายแนวคิดของควอซี-พรินซิเพ็ลลิอินเจคทีฟมอดูลและ (4) หาความสัมพันธ์ระหว่าง ควอซี-พรินซิเพ็ลลิอินเจคทีฟมอดูล ควอซี-สมอลพรินซิเพ็ลลิ-อินเจคทีฟมอดูล และโปรเจคทีฟมอดูล

กำหนดให้ R เป็นริง จะเรียก R-มอดูลทางขวา M ว่า *พรินซิเพ็ลลิอินเจคทีฟ* ก็ต่อเมื่อทุกๆ R-สาทิสสัณฐานจากอุดมคติมุขสำคัญทางขวาของ R ไปยัง M สามารถขยายไปยัง R-สาทิสสัณฐานจาก R ไปยัง M จะเรียก R-มอดูลทางขวา N ว่า *M-พรินซิเพ็ลลิอินเจคทีฟ* ก็ต่อเมื่อทุกๆ R-สาทิสสัณฐานจาก M-วัฏจักรมอดูลย่อยของ M ไปยัง N สามารถขยายไปยัง R-สาทิสสัณฐานจาก M ไปยัง N จะเรียก R-มอดูลทางขวา M ว่า *ควอซี-พรินซิเพ็ลลิอินเจคทีฟ* ก็ต่อเมื่อ M เป็น M-พรินซิเพ็ลลิอินเจคทีฟ เราขยายแนวคิดของ ควอซี-พรินซิเพ็ลลิอินเจคทีฟมอดูล มาเป็น ควอซี-สมอลพรินซิเพ็ลลิ-อินเจคทีฟมอดูล โดยจะเรียก R-มอดูลทางขวา N ว่า *M-สมอลพรินซิเพ็ลลิ-อินเจคทีฟ* ก็ต่อเมื่อทุกๆ R-สาทิสสัณฐานจากมอดูลย่อยแบบ M-วัฏจักรและสมอลของ M ไปยัง N สามารถขยายไปยัง R-สาทิสสัณฐานจาก M ไปยัง N จะเรียก R-มอดูลทางขวา M ว่า *ควอซี-สมอลพรินซิเพ็ลลิ-อินเจคทีฟ* ก็ต่อเมื่อ M เป็น M-สมอลพรินซิเพ็ลลิ-อินเจคทีฟ

ผลการวิจัยพบว่า (1) สำหรับโปรเจคทีฟมอดูล M จะได้ว่าเงื่อนไขดังต่อไปนี้มีความสมมูลกัน (a) ทุกๆมอดูลย่อยแบบ M-วัฏจักรและสมอลของ M เป็นโปรเจคทีฟ (b) ทุกๆมอดูลผลหารของมอดูลแบบ M-สมอล พรินซิเพ็ลลิ-อินเจคทีฟ เป็น M-สมอล พรินซิเพ็ลลิ-อินเจคทีฟ (c) ทุกๆมอดูลผลหารของอินเจคทีฟ R-มอดูล เป็น M-สมอล พรินซิเพ็ลลิ-อินเจคทีฟ (2) กำหนดให้ M เป็น R-มอดูลทางขวา และ $S = End_R(M)$ แล้วจะได้ว่าเงื่อนไขดังต่อไปนี้มีความสมมูลกัน (a) M เป็น ควอซี-

สมอลพรินซิเพิ็ลลิ-อินเจคทีฟ (b) $l_S(Ker(s)) = Ss$ สำหรับทุกๆ $s \in S$ โดยที่ $s(M) \ll M$ (c) $Ker(s) \subset Ker(t)$ โดยที่ $s, t \in S$ และ $s(M) \ll M$, แล้วจะได้ว่า $St \subset Ss$ (d) $l_S(Ker(s) \cap Im(t)) = l_S(Im(t)) + Ss$ สำหรับทุกๆ $s, t \in S$ โดยที่ $s(M) \ll M$ (3) กำหนดให้ M เป็นพรินซิเพิ็ลมอดูลซึ่งก่อกำเนิดตัวเอง ถ้า M เป็น ควอซี-สมอลพรินซิเพิ็ลลิ-อินเจคทีฟ แล้วจะได้ว่า (a) ถ้า $sS \oplus tS$ และ $Ss \oplus St$ เป็นผลบวกตรง โดยที่ $s, t \in J(S)$, แล้วจะได้ว่า $l_M(s) + l_M(t) = S$ (b) $l_S r_M(Ss) = Ss$ สำหรับแต่ละ $s \in J(S)$ (4) กำหนดให้ M เป็น ควอซี-สมอลพรินซิเพิ็ลลิ-อินเจคทีฟมอดูล โดยที่ $s, t \in S$ และ $s(M) \ll M$ (a) ถ้า $s(M)$ ฝังใน $t(M)$ แล้วจะได้ว่า Ss เป็นภาพของ St (b) ถ้า $t(M)$ เป็นภาพของ $s(M)$ แล้วจะได้ว่า St ฝังใน Ss (c) ถ้า $s(M)$ ไอโซมอร์ฟิก $t(M)$ แล้วจะได้ว่า Ss ไอโซมอร์ฟิก St

คำสำคัญ: มอดูลแบบควอซีพรินซิเพิ็ลลิ-อินเจคทีฟ มอดูลแบบควอซี-สมอลพรินซิเพิ็ลลิ-อินเจคทีฟ ริงอันตรสัณฐาน

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Passakorn Yordsorn

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List of Abbreviations

$A \oplus B$	A direct sum B
$End_R(M)$	The set of R-homomorphism from M to M
F	Field F
$f: M \to N$	A function f from M to N
$f(M)$, $Im(f)$	Image of f
$Hom_R(M,N)$	The set of R-homomorphism from M to N
$Ker(f)$	Kernel of f
$J(M)$, $Rad(M_R)$	Jacobson radical of a right R-module M
$J(R) = Rad(R_R)$	Jacobson radical of a ring R
$J(S)$	Jacobson radical of a ring S
$J(S) \subset {}_S S_S$	$J(S)$ is an (two-side) ideal of ring S
$l_M(A)$	Left annihilator of A in M
M_R	M is a right R-module
$M_1 \times M_2$	Cartesian products of M_1 and M_2
M/K	A factor module of M modulo K or a factor module of M by K
$M \cong N$	M isomorphic N
R	Ring R
R_R	Ring R is a right R-module is called Regular right R-module
$r_R(X)$	Right annihilator of X in R
$Z(M)$	Singular submodule of M
1_M	Identity map on a module M
$\begin{pmatrix} F & F \\ F & F \end{pmatrix} = M_2(F)$	The set of all 2×2 matrices having elements of a field F as entries

List of Abbreviations (Continued)

$\eta : M \to M/K$	η (*eta*) is the natural epimorphism of M onto M/K
$\iota = \iota_{A \subset B} : A \to B$	ι (*iota*) is the inclusion map of A in B
π_j	π_j is the j-th projection map
$\forall$	For all
$\cap$	Intersection of set
$\not\subset$	is not subset
$\subset$	subset
$\in$	is in, member of set
$\subset^e$	Essential (Large)
$\ll$	Superfluous (Small)
$\prod_{i \in I} N_i$	Direct product of N_i
$\bigoplus_{i=1}^{n} N_i$	Direct sum of N_i

CHAPTER 1

INTRODUCTION

In modules and rings theory research field, there are three methods for doing the research. Firstly, to study about the fundamental of algebra and modules theory over arbitrary rings. Secondly, to study about the modules over special rings. Thirdly, to study about ring R by way of the categories of R-modules. Many mathematicians have concentrated on these methods.

1.1 Background and Statement of the Problems

Many generalizations of the injectivity were obtained, e.g., *principally injectivity* and *mininjectivity*. In [2], V. Camillo introduced the definition of principally injective modules by calling a right R-module M is *principally injective* if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M.

In [7], Nicholson and Yousif studied to the structure of principally injective rings and gave some applications of these rings. A ring R is called *right principally injective* if every R-homomorphism from a principal right ideal of R to R can be extended to an R-homomorphism from R to R.

In [12], L.V. Thuyet, and T.C. Quynh introduced the definitions of a small principally module. A right R-module M is called *small principally injective* if every R-homomorphism from a small and principal right ideal aR to M can be extended to an R-homomorphism from R to M.

In [10], N. V. Sanh, K. P. Shum, S. Dhompongsa and S. Wongwai introduced the definitions of quasi principally injective modules. A right R-module M is called *quasi-principally injective* if every R-homomorphism from an M-cyclic submodule of M to M can be extended to M.

1.2 Purpose of the Study

In this thesis, we have the purposes of study which are to extend concept of the previous works and to generalize new concepts which are :

1.2.1 To extend the concept of *principally injective modules* [2].

1.2.2 To generalize the concept of *quasi principally injective modules* [10].

1.2.3 To establish and extend some new concepts which are dual to *quasi principally-injective modules* [10] and *quasi-small principally-injective modules*[19].

1.3 Research Questions and Hypothesis

We are interested in seeing to extend the characterizations and properties which remain valid from these previous concepts which can be extended from *principally injective modules* [2], *principally-injective rings* [7], *mininjective modules* [8], *principally quasi-injective modules* [9], *small principally quasi-injective modules* [18] and *quasi-small principally-injective modules* [19].

In this research, we introduce the definition of *quasi-small principally-injective modules* and give characterizations and properties of these modules which are extended from the previous works. By let M be a right R-module. A right R-module N is called *M-small principally injective* if every R-homomorphism from an M-cyclic small submodule of M to N can be extended to an R-homomorphism from M to N. Dually, a right R-module M is called *quasi-small P-injective* if it is M-small P-injective. Many of results in this research are extended from *principally injective rings* [7], *mininjective rings* [8], *small principally quasi-injective modules* [18] and *quasi-small principally-injective modules* [19].

1.4 Theoretical Perspective

In this thesis, we use many of the fundamental theories which are concerned to the rings and modules research. By the concerned theories are :

1.4.1 The fundamental of algebra theories.

1.4.2 The basic properties of rings and modules theory.

1.5 Delimitations and Limitations of the Study

For this thesis, we have the scopes and the limitations of studying which are concerned to the previous works which are:

1.5.1 To extend the concept of *M-small P-injective modules*.

1.5.2 To extend the concept of *quasi-small P-injective modules*.

1.5.3 To characterize the concept in 1.5.2 and find some new properties.

1.6 Significance of the Study

The advantage of education and studying in this research, we can improve and develop the concepts and knowledge in the algebra and modules research field.

CHAPTER 2

LITERATURE REVIEW

In this chapter we give notations, definitions and fundamental theories of the modules and rings theory which are used in this thesis.

2.1 Rings, Modules, Submodules and Endomorphism Rings

This section is assembled summary of various notations, terminology and some background theories which are concerned and used for this thesis.

2.1.1 Definition. [14] By a *ring* we mean a nonempty set R with two binary operations $+$ and $\cdot$, called *addition* and *multiplication* (also called *product*), respectively, such that

(1) $(R, +)$ is an additive abelian group.

(2) $(R, \cdot)$ is a multiplicative semigroup.

(3) Multiplication is distributive (on both sides) over addition; that is, for all $a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

The two distributive laws are respectively called the *left distributive* law and the *right distributive* law.

A *commutative ring* is a ring R in which multiplication is commutative; i.e. if $a \cdot b = b \cdot a$ for all $a, b \in R$. If a ring is not commutative it is called *noncommutative*.

A *ring with unity* is a ring R in which the multiplicative semigroup $(R, \cdot)$ has an identity element; that is, there exists $e \in R$ such that $ea = a = ae$ for all $a \in R$. The element e is called *unity* or the *identity* element of R. Generally, the unity or identity element is denoted by 1.

In this thesis, R will be an associative ring with identity.

2.1.2 Definition. [14] A nonempty subset I of a ring R is called an *ideal* of R if

(1) $a, b \in I$ implies $a - b \in I$.

(2) $a \in I$ and $r \in R$ imply $ar \in I$ and $ra \in I$.

2.1.3 Definition. [13] A subgroup I of $(R, +)$ is called a *left ideal* of R if $RI \subset I$, and a *right ideal* if $IR \subset I$.

2.1.4 Definition. [14] A right ideal I of a ring R is called *principal* if $I = aR$ for some $a \in R$.

2.1.5 Definition. [14] Let R be a ring, M an additive abelian group and $(m, r) \mapsto mr$, a mapping of $M \times R$ into M such that

(1) $mr \in M$

(2) $(m_1 + m_2)r = m_1r + m_2r$

(3) $m(r_1 + r_2) = mr_1 + mr_2$

(4) $(mr_1)r_2 = m(r_1r_2)$

(5) $m \cdot 1 = m$

for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. Then M is called a *right R-module*, often written as M_R.

Often mr is called the *scalar multiplication* or just *multiplication* of m by r on right. We define left R-module similarly.

2.1.6 Definition. [13] Let M be a right R-module. A subgroup N of $(M, +)$ is called a *submodule* of M if N is closed under multiplication with elements in R, that is $nr \in N$ for all $n \in N$, $r \in R$. Then N is also a right R-module by the operations induced from M :

$$N \times R \to N, (n, r) \mapsto nr, \text{ for all } n \in N, r \in R.$$

2.1.7 Proposition. *A subset N of an R-module M is a submodule of M if and only if*

(1) $0 \in N$.

(2) $n_1, n_2 \in N$ *implies* $n_1 - n_2 \in N$.

(3) $n \in N, r \in R$ *implies* $nr \in N$.

Proof. See [15, Lemma 5.3]. □

2.1.8 Definition. [1] Let M be a right R-module and let K be a submodule of M. Then the set of cosets

$$M/K = \{\ x + K \mid x \in M\ \}$$

is a right R-module relative to the addition and scalar multiplication defined via

$$(x + K) + (y + K) = (x + y) + K \quad \text{and} \quad (x + K)r = xr + K.$$

The additive identity and inverses are given by

$$K = 0 + K \quad \text{and} \quad -(x + K) = -x + K.$$

The module M/K is called (the *right R-factor module of*) *M modulo K* or the *factor module of M by K*.

2.1.9 Definition. [13] Let M and N be right R-modules. A function $f : M \longrightarrow N$ is called an (*R-module*) *homomorphism* if for all $m, m_1, m_2 \in M$ and $r \in R$

$$f(m_1 r + m_2) = f(m_1)r + f(m_2).$$

Equivalently, $f(m_1 + m_2) = f(m_1) + f(m_2)$ and $f(mr) = f(m)r$.

The set of R-homomorphisms of M in N is denoted by $Hom_R(M, N)$. In particular, with this addition and the composition of mappings, $Hom_R(M, M) = End_R(M)$ becomes a ring, called the *endomorphism ring* of M and $f \in End_R(M)$ is called *an R-endomorphism.* [13, 6.4]

2.1.10 Definition. [1] Let $f : M \longrightarrow N$ be an R-homomorphism. Then

(1) f is called *R-monomorphism* (or R-monic) if f is injective (one-to-one).

(2) f is called *R-epimorphism* (or R-epic) if f is surjective (onto).

(3) f is called *R-isomorphism* if f is bijective (one-to-one and onto).

Two modules M and N are said to be *R-isomorphic*, abbreviated $M \cong N$ in case there is an *R-isomorphism* $f : M \longrightarrow N$.

2.1.11 Definition. [1] Let K be a submodule of M. Then the mapping $\eta_K : M \longrightarrow M/K$ from M onto the factor module M/K defined by

$$\eta_K(x) = x + K \in M/K \qquad (x \in M)$$

is seen to be an R-epimorphism with kernel K. We call η_K the *natural epimorphism of M onto M/K.*

2.1.12 Definition. [1] Let $A \subset B$. Then the function $\iota = \iota_{A \subset B} : A \to B$ defined by $\iota = (1_{B|A}) : a \mapsto a$ for all $a \in A$ is called the *inclusion map* of A in B. Note that if $A \subset B$ and $A \subset C$, and if $B \neq C$, then $\iota_{A \subset B} \neq \iota_{A \subset C}$. Of course $1_A = \iota_{A \subset A}$.

2.1.13 Definition. [14] Let M and N be right R-modules and let $f : M \to N$ be an R-homomorphism. Then the set

$$Ker(f) = \{ x \in M \mid f(x) = 0 \}$$ is called the *kernel* of f

and

$$f(M) = \{ f(x) \in N \mid x \in M \}$$ is called the *homomorphic image* (or simply *image*) of M under f and is denoted by $Im(f)$.

2.1.14 Proposition. *Let M and N be right R-modules and let $f : M \to N$ be an R-homomorphism. Then*

(1) *$Ker(f)$ is a submodule of M.*

(2) *$Im(f) = f(M)$ is a submodule of N.*

Proof. See [13, 6.5]. □

2.1.15 Proposition. *Let M and N be right R-modules and let $f : M \to N$ be an R-isomorphism. Then the inverse mapping $f^{-1} : N \to M$ is an R-isomorphism.*

Proof. See [14, Chapter 14, 3]. □

2.1.16 Theorem. *Let M, M′, N and N′ be right R-modules and let $f : M \to N$ be an R-homomorphism.*

(1) *If $g : M \to M'$ is an epimorphism with $Ker(g) \subset Ker(f)$, then there exists a unique homomorphism $h : M' \to N$ such that*

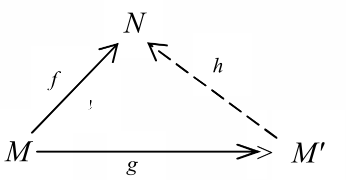
$$f = hg.$$

Moreover, $Ker(h) = g(Ker(f))$ *and* $Im(h) = Im(f)$, *so that h is monic if and only if* $Ker(g) = Ker(f)$ *and h is epic if and only if f is epic*.

(2) *If* $g : N' \to N$ *is a monomorphism with* $Im(f) \subset Im(g)$, *then there exists a unique homomorphism* $h : M \to N'$ *such that*

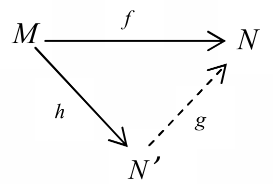
$$f = gh.$$

Moreover, $Ker(h) = Ker(f)$ *and* $Im(h) = g^{\leftarrow}(Im(f))$, *so that h is monic if and only if f is monic and h is epic if and only if* $Im(g) = Im(f)$.



$Ker(g) \subset Ker(f)$

(1)



(2)

Proof. See [1, Chapter 1, 46]. □

2.1.17 Definition. [20] A submodule K of the module M is fully invariant in M if $f(K) \subset K$ for every endomorphism f of M.

2.2 Essential and Superfluous Submodules

In this section, we give the definitions of essential and superfluous submodules and some theories which are used in this thesis.

2.2.1 Definition. [13] A submodule K of M is called *essential* (or *large*) in M, abbreviated $K \subset^{e} M$, if for every submodule L of M, $K \cap L = 0$ implies $L = 0$.

2.2.2 Definition. [13] A submodule K of M is called *superfluous* (or *small*) in M, abbreviated $K \ll M$, if for every submodule L of M, $K + L = M$ implies $L = M$.

2.2.3 Proposition. *Let M be a right R-module with submodules $K \subset N \subset M$ and $H \subset M$. Then*

(1) $N \ll M$ *if and only if* $K \ll M$ *and* $N/K \ll M/K$;

(2) $H + K \ll M$ *if and only if* $H \ll M$ *and* $K \ll M$.

Proof. See [1, Proposition 5.17]. □

2.2.4 Proposition. *If $K \ll M$ and $f : M \to N$ is a homomorphism then $f(K) \ll N$. In particular, if $K \ll M \subset N$ then $K \ll N$.*

Proof. See [1, Proposition 5.18]. □

2.3 Annihilators and Singular Modules

In this section, we give the definitions of annihilators, singular modules and some theories which are used in this thesis.

2.3.1 Definition. [1] Let M be a right (resp. left) R-module. For each $X \subset M$, the *right* (resp. *left*) *annihilator* of X in R is defined by

$$r_R(X) = \{ r \in R \mid xr = 0, \forall x \in X \} \text{ (resp. } l_R(X) = \{ r \in R \mid rx = 0, \forall x \in X \}).$$

For a singleton $\{x\}$, we usually abbreviated to $r_R(x)$ (resp. $l_R(x)$).

2.3.2 Proposition. *Let M be a right R-module, let X and Y be subsets of M and let A and B be subsets of R. Then*

(1) $r_R(X)$ *is a right ideal of* R.

(2) $X \subset Y$ *imples* $r_R(Y) \subset r_R(X)$.

(3) $A \subset B$ *imples* $l_M(B) \subset l_M(A)$.

(4) $X \subset l_M r_R(X)$ *and* $A \subset r_R l_M(A)$.

Proof. See [1, Proposition 2.14 and Proposition 2.15]. □

2.3.3 Proposition. *Let M and N be right R-modules and let $f : M \to N$ be a homomorphism. If N' is an essential submodule of N, then $f^{-1}(N')$ is an essential submodule of M.*

Proof. See [4, Lemma 5.8(a)]. □

2.3.4 Proposition. *Let M be a right R-module over an arbitrary ring R, the set*

$$Z(M) = \{\, x \in M \mid r_R(x) \text{ is essential in } R_R \,\}$$

is a submodule of M.

Proof. See [4, Lemma 5.9]. □

2.3.5 Definition. [4] The submodule $Z(M) = \{\, x \in M \mid r_R(x) \text{ is essential in } R_R \,\}$ is called the *singular submodule* of M. The module M is called a *singular module* if $Z(M) = M$. The module M is called *a nonsingular module* if $Z(M) = 0$.

2.4 Maximal and Minimal Submodules

In this section, we give the definitions and some properties of maximal submodules, minimal (or simple) submodules and some theories which are used in this thesis.

2.4.1 Definition. [13] A right R-module M is called *simple* if $M \neq 0$ and M has no submodules except 0 and M.

2.4.2 Definition. [13] A submodule K of M is called *maximal submodule* of M if $K \neq M$ and it is not properly contained in any proper submodules of M, i.e. K is *maximal in M* if, $K \neq M$ and for every $A \subset M$, $K \subset A$ implies $K = A$.

2.4.3 Definition. [13] A submodule N of M is called *minimal* (or *simple*) *submodule* of M if $N \neq 0$ and it has no non zero proper submodules of M, i.e. N is *minimal* (or *simple*) *in M* if $N \neq 0$ and for every nonzero submodules A of M, $A \subset N$ implies $A = N$.

2.4.4 Proposition. *Let M and N be right R-modules. If $f : M \to N$ is an epimorphism with $Ker(f) = K$, then there is a unique isomorphism $\sigma : M/K \to N$ such that $\sigma(m+K) = f(m)$*

for all $m \in M$.

Proof. See [1, Corollary 3.7]. □

2.4.5 Proposition. *Let K be a submodule of M. A factor module M/K is simple if and only if K is a maximal submodule of M.*

Proof. See [1, Corollary 2.10]. □

2.5 Injective and Projective Modules

In this section, we give the definitions of the injective modules, injective testing, projective modules and some theories which are used in this thesis.

2.5.1 Definition. [1] Let M be a right R-module. A right R-module U is called *injective relative to M* (or *U is M-injective*) if for every submodule K of M, for every homomorphism $\varphi : K \to U$ can be extended to a homomorphism $\alpha : M \to U$.

A right R-module U is said to be *injective* if it is M-injective for every right R-module M.

2.5.2 Proposition. *The following statements about a right R-module U are equivalent* :

(1) *U is injective*;

(2) *U is injective relative to R*;

(3) *For every right ideal $I \subset R_R$ and every homomorphism $h : I \to U$ there exists an $x \in U$ such that h is left multiplicative by x*

$$h(a) = xa \text{ for all } a \in I.$$

Proof. See [1, 18.3, Baer's Criterion]. □

2.5.3 Definition. [1] Let M be a right R-module. A right R-module U is called *projective relative to M* (or *U is M-projective*) if for every N_R, every epimorphism $g : M_R \to N_R$, for every homomorphism $\gamma : U_R \to N_R$ can be lifted to an R-homomorphism $\hat{\gamma} : U \to M$.

A right R-module U is said to be *projective* if it is projective for every right R-module M.

2.5.4 Proposition. *Every right* (*resp. left*) *R-module can be embedded in an injective right* (*resp. left*) *R-module.*

Proof. See [1, Proposition 18.6]. □

2.6 Direct Summands and Product of Modules

Given two modules M_1 and M_2 we can construct their Cartesian product $M_1 \times M_2$. The structure of this product module is then determined "co-ordinatewise" from the factors $M_1 \times M_2$. For this section we give the definitions of direct summand, the projection and the injection maps, product of modules and some theories which are used in this thesis.

2.6.1 Definition. [1] Let M be a right R-module. A submodule X of M is called a *direct summand* of M if there is a submodule Y of M such that $X \cap Y = 0$ and $X + Y = M$. We write $M = X \oplus Y$; such that Y is also a *direct summand*.

2.6.2 Definition. [1] Let M_1 and M_2 be R-modules. Then with their products module $M_1 \times M_2$ are associated the natural injections and projections

$$\varphi_j : M_j \to M_1 \times M_2 \qquad \text{and} \qquad \pi_j : M_1 \times M_2 \to M_j$$

($j = 1, 2$), are defined by

$$\varphi_1(x_1) = (x_1, 0), \qquad \varphi_2(x_2) = (0, x_2)$$

and

$$\pi_1(x_1, x_2) = x_1, \qquad \pi_2(x_1, x_2) = x_2.$$

Moreover, we have

$$\pi_1\varphi_1 = 1_{M_1} \qquad \text{and} \qquad \pi_2\varphi_2 = 1_{M_2}.$$

2.6.3 Definition. [1] Let A be a direct summand of M with complementary direct summand B, so $M = A \oplus B$. Then

$$\pi_A : a + b \mapsto a \qquad (a \in A, b \in B)$$

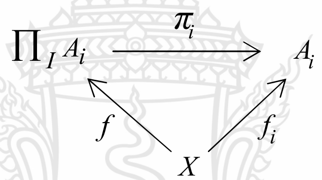
defines an epimorphism $\pi_A : M \to A$ is called *the projection of M on A along B*.

2.6.4 Definition. [13] Let $\{A_i\ ,\ i \in I\}$ be a family of objects in the category C. An object P in C with morphisms $\{\ \pi_i : P \to A_i\ \}$ is called the *product* of the family $\{A_i\ , i \in I\}$ if :

For every family of morphisms $\{\ f_i : X \to A_i\ \}$ in the category C, there is a unique morphism $f : X \to P$ with $\pi_i f = f_i$ for all $i \in I$.

For the object P, we usually write $\prod_{i \in I} A_i$, $\prod_I A_i$ or $\prod A_i$. If all A_i are equal to A, then we put $\prod_I A_i = A^I$.

The morphism π_i are called the *i-projections* of the product. The definition can be described by the following commutative diagram :



2.6.5 Definition. [13] Let $\{\ M_i\ , i \in I\ \}$ be a family of R-modules and $(\prod_{i \in I} M_i\ , \pi_i)$ the product of the M_i . For $m, n \in \prod_{i \in I} M_i$, $r \in R$, using

$$\pi_i (m + n) = \pi_i (m) + \pi_i (n) \quad \text{and} \quad \pi_i(mr) = \pi_i (m)r,$$

a right R-module structure is defined on $\prod_{i \in I} M_i$ such that the π_i are homomorphisms. With this structure $(\prod_{i \in I} M_i\ , \pi_i)$ is the product of the $\{\ M_i\ , i \in I\ \}$ in R-module.

2.6.6 Proposition. *Properties*:

(1) *If* $\{\ f_i : N \to M_i\ , i \in I\ \}$ *is a family of morphisms, then we get the map*

$$f : N \to \prod_{i \in I} M_i \qquad \textit{such that} \qquad n \mapsto (\ f_i(n))_{i \in I}$$

and $Ker(f) = \bigcap_I Ker(f_i)$ *since* $f(n) = 0$ *if and only if* $f_i(n) = 0$ *for all* $i \in I$.

(2) *For every $j \in I$, we have a canonical embedding*

$$\varepsilon_j : M_j \to \prod_{i \in I} M_i \,, \quad \textit{such that} \qquad m_j \mapsto (m_j \delta_{ji})_{i \in I} \,, m_j \in M_j \,,$$

with $\varepsilon_j \pi_j = 1_{M_j}$, *i.e.* π_j *is a retraction and* ε_j *a coretraction.*

This construction can be extended to larger subsets of I : *For a subset* $A \subset I$ *we form the product* $\prod_{i \in A} M_i$ *and a family of homomorphisms*

$$f_j : \prod_{i \in A} M_i \to M_j \,, \qquad f_j = \begin{cases} \pi_j \ \textit{for } j \in A, \\ 0 \ \textit{for } j \in I - A. \end{cases}$$

Then there is a unique homomorphism

$$\varepsilon_A : \prod_{i \in A} M_i \ \to \prod_{i \in I} M_i \ \text{ with } \ \varepsilon_A \pi_j = \begin{cases} \pi_j \ \textit{for } j \in A, \\ 0 \ \textit{for } j \in I - A. \end{cases}$$

The universal property of $\prod_{i \in A} M_i$ *yields a homomorphism*

$$\pi_A : \prod_{i \in I} M_i \to \prod_{i \in A} M_i \ \textit{with } \pi_A \pi_j = \pi_j \text{ for } j \in I.$$

Together this implies $\varepsilon_A \pi_A \pi_j = \varepsilon_A \pi_j = \pi_j$ *for all* $j \in I$, *and by the properties of the product* $\prod_{i \in A} M_i$,

we get $\varepsilon_A \pi_A = 1_{M_A}$.

Proof. See [13, 9.3, Properties (1), (2)] □

2.6.7 Definition. [1] We say $(M_\alpha)_{\alpha \in A}$ is independent in case for each $\alpha \in A$

$$M_\alpha \cap (\sum_{\beta \neq \alpha} M_\beta) = 0.$$

If the submodules $(M_\alpha)_{\alpha \in A}$ of M are independent, we say that the sum $\sum_A M_\alpha$ is *direct* and write

$$\sum_A M_\alpha = \bigoplus_A M_\alpha \,.$$

2.6.8 Proposition. [1] *Let* $(M_\alpha)_{\alpha \in A}$ *be an indexed set of submodules of a module* M *with inclusion maps* $(i_\alpha)_{\alpha \in A}$. *Then the following are equivalent*:

(a) $\sum_A M_\alpha$ *is the internal direct sum of* $(M_\alpha)_{\alpha \in A}$;

(b) $i = \bigoplus_A i_\alpha : \bigoplus_A M_\alpha \to M$ *is monic*;

(c) $(M_\alpha)_{\alpha \in A}$ *is independent*;

(d) $(M_\alpha)_{\alpha \in F}$ *is independent for every finite subset* $F \subset A$;

(e) *For every pair* $B, C \subset A$, *if* $B \cap C = \varnothing$, *then*

$$(\sum_B M_\beta) \cap (\sum_C M_\gamma) = 0.$$

Proof. See [1, Proposition 6.10]. □

2.7 Generated and Cogenerated Classes

In this section, we give some definitions and theories of the generated and cogenerated classes which are concerned in this thesis.

2.7.1 Definition. [13] A subset *X* of a right *R*-module *M* is called a *generating set* of *M* if $XR = M$. We also say that *X generates M* or *M* is *generated by X*. If there is a finite generating set in *M*, then *M* is called *finitely generated*.

2.7.2 Definition. [1] Let $\mathcal{U}$ be a class of right *R*-modules. A module *M* is (*finitely*) *generated by* $\mathcal{U}$ (or $\mathcal{U}$ (*finitely*) *generates M*) if there exists an epimorphism

$$\bigoplus_{i \in I} U_i \longrightarrow M$$

for some (finite) set *I* and $U_i \in \mathcal{U}$ for every $i \in I$.

If $\mathcal{U} = \{U\}$ is a singleton, then we say that *M* is (*finitely*) *generated by* U or (*finitely*) *U-generates*; this means that there exists an epimorphism

$$U^{(I)} \longrightarrow M$$

for some (finite) set *I*.

2.7.3 Proposition. *If a module M has a generating set* $L \subset M$, *then there exists an epimorphism*

$$R^{(L)} \longrightarrow M$$

Moreover, *M is finitely R-generated if and only if M is finitely generated*.

Proof. See [1, Theorem 8.1]. □

2.7.4 Definition. [17] Let M be a right R-module. A submodule N of M is said to be an M-*cyclic submodule* of M if it is the image of an endomorphism of M.

2.7.5 Definition. [1] Let $\mathcal{U}$ be a class of right R-modules. A module M is (*finitely*) *cogenerated by* $\mathcal{U}$ (or $\mathcal{U}$ (*finitely*) *cogenerates* M) if there exists a monomorphism

$$M \to \prod_{i \in I} U_i$$

for some (finite) set I and $U_i \in \mathcal{U}$ for every $i \in I$.

If $\mathcal{U} = \{U\}$ is a singleton, then we say that a module M is (*finitely*) *cogenerated by* $\mathcal{U}$ or (*finitely*) U-*cogenerates*; this means that there exists a monomorphism

$$M \to U^I$$

for some (finite) set I.

2.8 The Trace and Reject

In this section, we give some definitions and theories of the trace and reject which are concerned in this thesis.

2.8.1 Definition. [1] Let $\mathcal{U}$ be a class of right R-modules. The *trace* of $\mathcal{U}$ in M and the *reject* of $\mathcal{U}$ in M are defined by

$$Tr_M(\mathcal{U}) = \sum \{ Im(h) \mid h : U \to M \text{ for some } U \in \mathcal{U} \}$$

and

$$Rej_M(\mathcal{U}) = \bigcap \{ Ker(h) \mid h : M \to U \text{ for some } U \in \mathcal{U} \}.$$

If $\mathcal{U} = \{U\}$ is a singleton, then the trace of $\mathcal{U}$ in M and the reject of $\mathcal{U}$ in M are in the form

$$Tr_M(U) = \sum \{ Im(h) \mid h \in Hom_R(U, M) \}$$

and

$$Rej_M(U) = \bigcap \{ Ker(h) \mid h \in Hom_R(M, U) \}.$$

2.8.2 Proposition. *Let $\mathcal{U}$ be a class of right R-modules and let M be a right R-module. Then*

(1) $Tr_M(\mathfrak{U})$ *is the unique largest submodule L of M generated by* $\mathfrak{U}$;

(2) $Rej_M(\mathfrak{U})$ *is the unique smallest submodule K of M such that* M/K *is cogenerated by* $\mathfrak{U}$.

Proof. See [1, Proposition 8.12]. □

2.9 Socle and Radical of Modules

In this section, we give some definitions and theories of the socle and radical of modules which are used in this thesis.

2.9.1 Definition. [13] Let *M* be a right *R*-module. The *socle* of *M*, $Soc(M)$, we denote the sum of all simple submodules of *M*. If there are no simple submodules in *M* we put $Soc(M) = 0$.

2.9.2 Definition. [13] Let *M* be a right *R*-module. The *radical* of *M*, $Rad(M)$, we denote the intersection of all maximal submodules of *M*. If *M* has no maximal submodules we set $Rad(M) = M$.

2.9.3 Proposition. *Let* $\mathcal{E}$ *be the class of simple R-modules and let M be an R-module. Then*

$$\begin{aligned} Soc(M) &= Tr_M(\mathcal{E}) \\ &= \bigcap \{ L \subset M \mid L \text{ is essential in } M \}. \end{aligned}$$

Proof. See [13, 21.1]. □

2.9.4 Proposition. *Let* $\mathcal{E}$ *be the class of simple R-modules and let M be an R-module. Then*

$$\begin{aligned} Rad(M) &= Rej_M(\mathcal{E}) \\ &= \sum \{ L \subset M \mid L \text{ is superfluous in } M \}. \end{aligned}$$

Proof. See [13, 21.5]. □

2.9.5 Proposition. *Let M be a right R-module. A right R-module M is finitely generated if and only if* $Rad(M) \ll M$ *and* $M/Rad(M)$ *is finitely generated.*

Proof. See [13, 21.6, (4)]. □

2.9.6 Proposition. *Let M be a right R-module. Then* $Soc(M) \subset^{e} M$ *if and only if every non-zero submodule of M contains a minimal submodule.*

Proof. See [1, Corollary 9.10]. □

2.10 The Radical of a Ring

In this section, we give some definitions and theories of the radical of a ring which are used in this thesis.

2.10.1 Definition. [1] Let R be a ring. The radical $Rad(R_R)$ of R_R is an (two side) ideal of R. This ideal of R is called the (*Jacobson*) *radical* of R, and we usually abbreviated by

$$J(R) = Rad(R_R).$$

Since $R = 1R$ is finite generated, $J(R) \ll R$. If $a \in J(R)$, then $aR \subset J(R) \ll R$ so $aR \ll R$. If $aR \ll R$, then $aR \subset J(R)$ and so $a \in aR \subset J(R)$. This shows that $a \in J(R)$ if and only if $aR \ll R$.

2.10.2 Definition. [1] Let R be a ring. An element $x \in R$ is called *right* (*left*) *quasi-regular* if $1-x$ has a right (resp. left) inverse in R.

An element $x \in R$ is called *quasi-regular* if it is right and left quasi-regular.

A subset of R is said to be (*right*, *left*) *quasi-regular* if every element in it has the corresponding property.

2.10.3 Proposition. *Given a ring R for each of the following subsets of R is equal to the radical J(R) of R.*

(J_1) *The intersection of all maximal right (left) ideals of R;*

(J_2) *The intersection of all right (left) primitive ideals of R;*

(J_3) $\{ x \in R \mid rxs \text{ is quasi-regular for all } r, s \in R \}$;

(J_4) $\{ x \in R \mid rx \text{ is quasi-regular for all } r \in R \}$;

(J_5) $\{ x \in R \mid xs \text{ is quasi-regular for all } s \in R \}$;

(J_6) *The union of all the quasi-regular right (left) ideals of R*;

(J_7) *The union of all the quasi-regular ideals of R*;

(J_8) *The unique largest superfluous right (left) ideals of R*;

Moreover, (J_3), (J_4), (J_5), (J_6) *and* (J_7) *also describe the radical* $J(R)$ *if* "*quasi-regular*" *is replaced by* "*right quasi-regular*" *or by* "*left quasi-regular*".

Proof. See [1, Theorem 15.3]. □

2.10.4 Proposition. *Let R be a ring with radical* $J(R)$. *Then for every right R-module M*,

$$J(R)M_R \subset Rad(M_R).$$

If R is semisimple modulo its radical, then for every right R-module,

$$J(R)M_R = Rad(M_R)$$

and $M/J(R)M_R$ *is semisimple.*

Proof. See [1, Corollary 15.18]. □

CHAPTER 3

RESEARCH RESULT

In this chapter, we present the results of M-small P-injective modules and quasi-small P-injective modules.

3.1 *M*-small *P*-injective Modules

3.1.1 Definition. Let M be a right R-module. A right R-module N is called *M-small principally injective* (briefly, *M-small P-injective*) if every R-homomorphism from M-cyclic small submodule of M to N can be extended to an R-homomorphism from M to N. Equivalently, for any endomorphism s of M with $s(M) \ll M$, every R-homomorphism from $s(M)$ to N can be extended to an R-homomorphism from M to N.

3.1.2 Lemma. *Let M and N be right R-modules. Then N is M-small P-injective if and only if for each $s \in S = End_R(M)$ with $s(M) \ll M$,*

$$Hom_R(M, N)s = \{f \in Hom_R(M, N) : f(Ker(s)) = 0\}.$$

Proof. ($\Rightarrow$) Assume that N is M-small P-injective. Let $s \in S = End_R(M)$ and $s(M) \ll M$. To show that $Hom_R(M, N)s = \{f \in Hom_R(M, N) : f(Ker(s)) = 0\}$. ($\subset$) Let $gs \in Hom_R(M, N)s$. Since $s : M \to M$ and $g : M \to N$, $gs : M \to N$. Let $x \in Ker(s)$. Then $gs(x) = g(s(x)) = g(0) = 0$. Hence $gs \in \{f \in Hom_R(M, N) : f(Ker(s)) = 0\}$. This shows that $Hom_R(M, N)s \subset \{f \in Hom_R(M, N) : f(Ker(s)) = 0\}$. ($\supset$) Let $f \in \{ f \in Hom_R(M, N) : f(Ker(s)) = 0\}$. Let $x \in Ker(s)$. Since $f(Ker(s)) = 0$, $f(x) = 0$. Then $Ker(s) \subset Ker(f)$. By Proposition 2.1.16, there exists an R-homomorphism $\varphi : s(M) \to N$ such that $f = \varphi s$. Since $s(M) \ll M$, there exists an R-homomorphism $\hat{\varphi} : M \to N$ such that $\varphi = \hat{\varphi}\iota$ where $\iota : s(M) \to M$ is the inclusion map.

Hence $f = \varphi s = (\hat{\varphi}\iota)s = \hat{\varphi}s \in Hom_R(M,N)s$.

($\Leftarrow$) Let $s \in S = End_R(M)$ with $s(M) \ll M$ and $\varphi : s(M) \to N$ be an R-homomorphism. Then $\varphi s \in Hom_R(M,N)$. Let $x \in Ker(s)$. Then $\varphi s(x) = \varphi(0) = 0$. Therefore $\varphi s(Ker(s)) = 0$. Then by assumption, $\varphi s \in Hom_R(M,N)s$. Hence $\varphi s = \mu s$, for some $\mu \in Hom_R(M,N)$. This shows that N is M-small P-injective. □

3.1.3 Example. *Let* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ *where* F *is a field,* $M_R = R_R$ *and* $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. *Then* N *is* M*-small* P*-injective.*

Proof. We have only $X_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$, $X_3 = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $X_4 = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$, $X_5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $X_6 = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ are nonzero submodules of M, and we see that only $X_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ is only a small submodule of M because for every $X_i \subset M$, $2 \le i \le 5$, $X_i \ne M$ then $X_1 + X_i \ne M$. Now we show that X_1 is an M-cyclic submodule of M. Define $s : \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} \to \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ by $s\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ for every $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. To show that s is well-defined. Let $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$, $\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ such that $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$. Then $s\left(\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\right) = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix} = s\left(\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right)$. To show that s is an R-homomorphism. Let $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$, $\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and $\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \in R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then $s\left(\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right) = s\left(\begin{pmatrix} a_1r_1 & a_1r_2+b_1r_3 \\ 0 & c_1r_3 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right) = s\left(\begin{pmatrix} a_1r_1+a_2 & a_1r_2+b_1r_3+b_2 \\ 0 & c_1r_3+c_2 \end{pmatrix}\right) = \begin{pmatrix} 0 & a_1r_2+b_1r_3+b_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1r_2+b_1r_3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix} = s\left(\begin{pmatrix} a_1r_1 & a_1r_2+b_1r_3 \\ 0 & c_1r_3 \end{pmatrix}\right) + s\left(\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right) = s\left(\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix}\right) + s\left(\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\right)$. We must show that s is an R-epimorphism. Let $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = X_1$. Then there exists $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ such that $s\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. Let $\varphi : X_1 \to N$ be an R-homomorphism. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in X_1$, there exists x_{11}, $x_{12} \in F$ such that $\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$. Then $\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) =$

$\varphi\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}x_{11}&x_{12}\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&x_{12}\\0&0\end{pmatrix}$. Then $\begin{pmatrix}x_{11}&x_{12}\\0&0\end{pmatrix} = \begin{pmatrix}0&x_{12}\\0&0\end{pmatrix}$ so $x_{11} = 0$. Define $\hat{\varphi}: M \to N$ by $\hat{\varphi}\left(\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix}\right) = \begin{pmatrix}x_{12}a_{11}&x_{12}a_{12}\\0&0\end{pmatrix}$ for every $\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix} \in M$. To show that $\hat{\varphi}$ is well-defined. Let $\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix}, \begin{pmatrix}b_{11}&b_{12}\\0&b_{22}\end{pmatrix} \in M$ such that $\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix} = \begin{pmatrix}b_{11}&b_{12}\\0&b_{22}\end{pmatrix}$. Then $\hat{\varphi}\left(\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix}\right) = \begin{pmatrix}x_{12}a_{11}&x_{12}a_{12}\\0&0\end{pmatrix} = \begin{pmatrix}x_{12}b_{11}&x_{12}b_{12}\\0&0\end{pmatrix} = \hat{\varphi}\left(\begin{pmatrix}b_{11}&b_{12}\\0&b_{22}\end{pmatrix}\right)$. To show that $\hat{\varphi}$ is an R-homomorphism. Let $\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix}, \begin{pmatrix}b_{11}&b_{12}\\0&b_{22}\end{pmatrix} \in \begin{pmatrix}F&F\\0&F\end{pmatrix}$ and $\begin{pmatrix}r_1&r_2\\0&r_3\end{pmatrix} \in R$. Then $\hat{\varphi}\left(\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix}\begin{pmatrix}r_1&r_2\\0&r_3\end{pmatrix} + \begin{pmatrix}b_{11}&b_{12}\\0&b_{22}\end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix}a_{11}r_1&a_{11}r_2+a_{12}r_3\\0&a_{22}r_3\end{pmatrix} + \begin{pmatrix}b_{11}&b_{12}\\0&b_{22}\end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix}a_{11}r_1+b_{11}&a_{11}r_2+a_{12}r_3+b_{12}\\0&a_{22}r_3+b_{22}\end{pmatrix}\right) = \begin{pmatrix}x_{12}(a_{11}r_1+b_{11})&x_{12}(a_{22}r_3+b_{22})\\0&0\end{pmatrix} = \begin{pmatrix}x_{12}a_{11}r_1+x_{12}b_{11}&x_{12}a_{22}r_3+x_{12}b_{22}\\0&0\end{pmatrix} = \begin{pmatrix}x_{12}a_{11}r_1&x_{12}a_{22}r_3\\0&0\end{pmatrix} + \begin{pmatrix}x_{12}b_{11}&x_{12}b_{22}\\0&0\end{pmatrix} = \hat{\varphi}\left(\begin{pmatrix}a_{11}r_1&a_{11}r_2+a_{12}r_3\\0&a_{22}r_3\end{pmatrix}\right) + \hat{\varphi}\left(\begin{pmatrix}b_{11}&b_{12}\\0&b_{22}\end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix}\begin{pmatrix}r_1&r_2\\0&r_3\end{pmatrix}\right) + \hat{\varphi}\left(\begin{pmatrix}b_{11}&b_{12}\\0&b_{22}\end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix}a_{11}&a_{12}\\0&a_{22}\end{pmatrix}\right)\begin{pmatrix}r_1&r_2\\0&r_3\end{pmatrix} + \hat{\varphi}\left(\begin{pmatrix}b_{11}&b_{12}\\0&b_{22}\end{pmatrix}\right)$. To show that $\hat{\varphi}\iota = \varphi$. Let $\begin{pmatrix}0&x\\0&0\end{pmatrix} \in X_1$. Then $\hat{\varphi}\iota\left(\begin{pmatrix}0&x\\0&0\end{pmatrix}\right) = \hat{\varphi}\left(\iota\begin{pmatrix}0&x\\0&0\end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix}0&x\\0&0\end{pmatrix}\right) = \begin{pmatrix}0&x_{12}x\\0&0\end{pmatrix} = \begin{pmatrix}x_{11}&x_{12}\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&x\end{pmatrix} = \varphi\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\begin{pmatrix}0&0\\0&x\end{pmatrix} = \varphi\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&x\end{pmatrix}\right) = \varphi\left(\begin{pmatrix}0&x\\0&0\end{pmatrix}\right)$. This shows that $\hat{\varphi}$ is an extension of φ. Thus N is M-small P-injective. □

3.1.4 Proposition. *Let M be a right R-modules and $\{N_i, i \in I\}$ be a family of right R-modules. Then the direct product $\prod_{i\in I} N_i$ is M-small P-injective if and only if each N_i is M-small P-injective.*

Proof. ($\Rightarrow$) Let $\{N_i, i \in I\}$ be a family of right R-modules and the direct product $\prod_{i\in I} N_i$ is M-small P-injective. Let $i \in I$, we must show that N_i is M-small P-injective. Let $s \in S = End_R(M)$ with $s(M) \ll M$ and let $\varphi: s(M) \to N_i$ be an R-homomorphism. Let π_i and φ_i, for each $i \in I$, be the i-th projection map and the i-th injection map, respectively.

Since $\prod_{i\in I} N_i$ is M-small P-injective, there exists an R-homomorphism $\hat{\varphi}: M \to \prod_{i\in I} N_i$ such that $\hat{\varphi}\,\iota = \varphi_i\varphi$ where $\iota : s(M) \to M$ is the inclusion map. Thus $\pi_i\hat{\varphi}\,\iota = \pi_i\varphi_i\varphi$, so by Definition 2.6.2, $\pi_i\hat{\varphi}\,\iota = \varphi$. Thus $\pi_i\hat{\varphi}$ is an extension of φ.

($\Leftarrow$) Let N_i be M-small P-injective for each $i \in I$. To show that $\prod_{i\in I} N_i$ is M-small P-injective. Let $s \in S = End_R(M)$ with $s(M) \ll M$ and let $\varphi : s(M) \to \prod_{i\in I} N_i$ be an R-homomorphism. Let π_i be the i-th projection map. Since, for each i, N_i is M-small P-injective, there exists an R-homomorphism $\alpha_i : M \to N_i$ such that $\pi_i\varphi = \alpha_i\iota$ where $\iota : s(M) \to M$ is the inclusion map. Then by Definition 2.6.5 and Proposition 2.6.6, we obtain $\hat{\varphi} : M \to \prod_{i\in I} N_i$ such that $\pi_i\hat{\varphi} = \alpha_i$ for each $i \in I$. Then $\pi_i\hat{\varphi}\,\iota = \alpha_i\iota$, so $\pi_i\varphi = \alpha_i\iota = \pi_i\hat{\varphi}\,\iota$. Hence $\pi_i\varphi = \pi_i\hat{\varphi}\,\iota$ for each $i \in I$. Therefore $\varphi = \hat{\varphi}\,\iota$. □

3.1.5 Lemma. *Let M and N_i $(1 \le i \le n)$ be right R-modules. Then $\bigoplus_{i=1}^{n} N_i$ is M-small P-injective if and only if N_i is M-small P-injective for each $i = 1, 2, 3, \dots, n$.*

Proof. ($\Rightarrow$) Let $i \in \{1, 2, 3, \dots, n\}$. To show that N_i is M-small P-injective. Let $s \in S = End_R(M)$ with $s(M) \ll M$ and let $\varphi : s(M) \to N_i$ be an R-homomorphism. Let π_i and φ_i be the i-th projection map and the i-th injection map, respectively. Since $\bigoplus_{i=1}^{n} N_i$ is M-small P-injective, there exists an R-homomorphism $\hat{\varphi} : M \to \bigoplus_{i=1}^{n} N_i$ such that $\hat{\varphi}\,\iota = \varphi_i\varphi$ where $\iota : s(M) \to M$ is the inclusion map. Thus $\pi_i\hat{\varphi}\,\iota = \pi_i\varphi_i\varphi$, so by Definition 2.6.2, $\pi_i\hat{\varphi}\,\iota = \varphi$. Thus $\pi_i\hat{\varphi}$ is an extension of φ.

($\Leftarrow$) We must show that $\bigoplus_{i=1}^{n} N_i$ is M-small P-injective. Let $s \in S = End_R(M)$ with $s(M) \ll M$ and let $\alpha : s(M) \to \bigoplus_{i=1}^{n} N_i$ be an R-homomorphism. Since for each $i \in \{1, 2, 3, \dots, n\}$, N_i is M-small P-injective, there exists an R-homomorphism $\alpha_i : M \to N_i$ such that $\alpha_i\iota = \pi_i\alpha$ where π_i is the i-th projection map from $\bigoplus_{i=1}^{n} N_i$ to N_i and $\iota : s(M) \to M$ is the inclusion map. Set $\hat{\alpha} = \iota_1\alpha_1 + \iota_2\alpha_2 + \dots + \iota_n\alpha_n : M \to \bigoplus_{i=1}^{n} N_i$ where $\iota_i : N_i \to \bigoplus_{i=1}^{n} N_i$

for each $i \in \{1, 2, 3, \dots, n\}$ is the i-injection map. We must to show that $\hat{\alpha}$ is an extension of α. Let $s(m) \in s(M)$. Then $\hat{\alpha}\,\iota(s(m)) = \hat{\alpha}\,(s(m)) = \iota_1\alpha_1(s(m)) + \iota_2\alpha_2(s(m)) + \dots + \iota_n\alpha_n(s(m)) = \alpha_1(s(m)) + \alpha_2(s(m)) + \dots + \alpha_n(s(m)) = \alpha_1\iota_1(s(m)) + \alpha_2\iota_2(s(m)) + \dots + \alpha_n\iota_n(s(m)) = \pi_1\alpha(s(m)) + \pi_2\alpha(s(m)) + \dots + \pi_n\alpha(s(m)) = (\pi_1 + \pi_2 + \dots + \pi_n)\alpha(s(m)) = \alpha(s(m))$. Then $\bigoplus_{i=1}^{n} N_i$ is M-small P-injective. □

3.1.6 Lemma. *Any direct summand of an M-small P-injective module is again M-small P-injective.*

Proof. Let N be an M-small P-injective module and let A be a direct summand of N. To show that A is an M-small P-injective. Let $s \in S = End_R(M)$ with $s(M) \ll M$ and let $\alpha : s(M) \to A$ be an R-homomorphism. Since N is M-small P-injective, there exists an R-homomorphism $\hat{\alpha} : M \to N$ such that $\varphi\alpha = \hat{\alpha}\,\iota$ where $\iota : s(M) \to M$ is the inclusion map and $\varphi : A \to N$ is the injection map. Let $\pi : N \to A$ be the projection map. Then $\pi\varphi\alpha = \pi\,\hat{\alpha}\,\iota$. Hence by Definition 2.6.2, $\alpha = \pi\hat{\alpha}\,\iota$. Then $\pi\hat{\alpha}$ is an extension of α. □

3.1.7 Theorem. *The following conditions are equivalent for a projective module M.*

(1) *Every M-cyclic small submodule of M is projective.*

(2) *Every factor module of an M-small P-injective module is M-small P-injective.*

(3) *Every factor module of an injective R-module is M-small P-injective.*

Proof. (1) $\Rightarrow$ (2) Let N be an M-small P-injective module, X a submodule of N. To show that N/X is an M-small P-injective. Let $s \in S = End_R(M)$ with $s(M) \ll M$ and let $\alpha : s(M) \to N/X$ be an R-homomorphism. Since $s(M)$ is projective, there exists an R-homomorphism $\varphi : s(M) \to N$ such that $\alpha = \eta\varphi$ where $\eta : N \to N/X$ is the natural R-epimorphism. Since N is M-small P-injective, there exists an R-homomorphism $\beta : M \to N$ such that $\varphi = \beta\iota$ where $\iota : s(M) \to M$ is the inclusion map. Then $\alpha = \eta\varphi = \eta\beta\iota$. Hence $\alpha = \eta\beta\iota$. Therefore $\eta\beta$ is an extension of α. Thus N/X is an M-small P-injective.

(2) $\Rightarrow$ (3) Let N be an injective R-module and X be a submodule of N. It is clear that an injective R-module is an M-small P-injective module, so N is M-small P-injective. Then by (2), N/X is an M-small P-injective.

(3) $\Rightarrow$ (1) Let $s(M) \ll M$, $\gamma : A \to B$ be an R-epimorphism and let $\varphi : s(M) \to B$ be an R-homomorphism. Let E be an injective R-module and embed A in E by Proposition 2.5.4. Since γ is an R-epimorphism, by Proposition 2.4.4, there exists an R-isomorphism $\sigma : A/Ker(\gamma) \to B$ such that $\gamma = \sigma\eta_1$ where $\eta_1 : A \to A/Ker(\gamma)$ is the natural R-epimorphism. Then by Proposition 2.1.15, we have $\sigma^{-1} : B \to A/Ker(\gamma)$ is an R-isomorphism, so $B \cong A/Ker(\gamma)$ and $A/Ker(\gamma)$ is a submodule of $E/Ker(\gamma)$. By assumption, there exists an R-homomorphism $\hat{\varphi} : M \to E/Ker(\gamma)$ such that $\iota_1\sigma^{-1}\varphi = \hat{\varphi}\iota_2$ where $\iota_1 : A/Ker(\gamma) \to E/Ker(\gamma)$ and $\iota_2 : s(M) \to M$ are the inclusion maps. Since M is projective, there exists an R-homomorphism $\beta : M \to E$ such that $\hat{\varphi} = \eta_2\beta$ where $\eta_2 : E \to E/Ker(\gamma)$ is the natural R-epimorphism. Then $\hat{\varphi}\iota_2 = \eta_2\beta\iota_2$. Hence $\iota_1\sigma^{-1}\varphi = \hat{\varphi}\iota_2 = \eta_2\beta\iota_2$. It follows that $\iota_1\sigma^{-1}\varphi = \eta_2\beta\iota_2$. To show that $\beta(s(M)) \subset A$. Let $s(m) \in s(M)$. Then $\iota_1\sigma^{-1}\varphi(s(m)) = \eta_2\beta\iota_2(s(m)) = \eta_2\beta(s(m)) = \eta_2(\beta(s(m))) = \beta(s(m)) + Ker(\gamma)$. Hence $\iota_1\sigma^{-1}\varphi(s(m)) = \sigma^{-1}\varphi(s(m)) = a + Ker(\gamma)$ for some $a \in A$, so $\beta(s(m)) + Ker(\gamma) = a + Ker(\gamma)$. Thus $\beta(s(m)) - a \in Ker(\gamma)$. It follows that $\beta(s(m)) = (\beta(s(m)) - a) + a \in Ker(\gamma) + A = A$. To show that $\varphi = \gamma\beta$. Let $s(m) \in s(M)$. Then $\iota_1\sigma^{-1}\varphi(s(m)) = \sigma^{-1}\varphi(s(m)) = \eta_2\beta\iota_2(s(m)) = \eta_2\beta(s(m))$. Hence $\iota_1\sigma^{-1}\varphi(s(m)) = \eta_2\beta(s(m)) = \beta(s(m)) + Ker(\gamma)$, so $\iota_1\sigma^{-1}\varphi(s(m)) = \beta(s(m)) + Ker(\gamma)$. Since γ is an R-epimorphism, $\varphi(s(m)) = \gamma(a)$ for some $a \in A$. Thus $\iota_1\sigma^{-1}\varphi(s(m)) = \iota_1\sigma^{-1}\gamma(a) = \sigma^{-1}\gamma(a) = \eta_1(a) = a + Ker(\gamma)$. It follows that $\beta(s(m)) + Ker(\gamma) = a + Ker(\gamma)$. Then $\beta(s(m)) - a \in Ker(\gamma)$. Hence $\gamma(\beta(s(m)) - a) = 0$, so $\gamma\beta(s(m)) = \gamma(a) = \varphi(s(m))$. Thus $\gamma\beta(s(m)) = \varphi(s(m))$. This shows that β lifts φ. □

3.2 Quasi-small *P*-injective Modules

A right R-module M is called *quasi-small P-injective* if it is M-small P-injective. In this section, we present the results of characterizations and properties of the endomorphism ring of quasi-small P-injective modules.

3.2.1 Lemma. *Let M be a right R-module and $S = End_R(M)$. Then the following conditions are equivalent :*

(1) *M is quasi-small P-injective.*

(2) $l_S(Ker(s)) = Ss$ *for all* $s \in S$ *with* $s(M) \ll M$.

(3) $Ker(s) \subset Ker(t)$, *where* $s, t \in S$ *with* $s(M) \ll M$, *implies* $St \subset Ss$.

(4) $l_S(Ker(s) \cap Im(t)) = l_S(Im(t)) + Ss$ *for all* $s, t \in S$ *with* $s(M) \ll M$.

Proof. (1) $\Rightarrow$ (2) Let $s \in S = End_R(M)$ with $s(M) \ll M$. ($\supset$) Let $fs \in Ss$. To show that $fs \in l_S(Ker(s))$. Let $x \in Ker(s)$. Then $s(x) = 0$, $fs(x) = f(s(x)) = f(0) = 0$. ($\subset$) Let $f \in l_S(Ker(s))$. To show that $f \in Ss$. Let $x \in Ker(s)$. Since $f(Ker(s)) = 0$, $f(x) = 0$. Then $x \in Ker(f)$. This shows that $Ker(s) \subset Ker(f)$. Since $s : M \to s(M)$ is an R-epimorphism, by Proposition 2.1.16, there exists an R-homomorphism $\varphi : s(M) \to M$ such that $f = \varphi s$. Since $s(M) \ll M$ and M is quasi-small P-injective, there exists an R-homomorphism $\hat{\varphi} : M \to M$ such that $\varphi = \hat{\varphi}\iota$ where $\iota : s(M) \to M$ is the inclusion map. Hence $f = \varphi s = (\hat{\varphi}\iota)s = \hat{\varphi}s \in Ss$. This shows that $f \in Ss$.

(2) $\Rightarrow$ (1) To show that M is quasi-small P-injective. Let $s \in S = End_R(M)$ with $s(M) \ll M$ and let $\varphi : s(M) \to M$ be an R-homomorphism. Then $\varphi s \in S$. To show that $\varphi s \in l_S(Ker(s))$. Let $x \in Ker(s)$. Then $s(x) = 0$ so $\varphi s(x) = \varphi(s(x)) = \varphi(0) = 0$. This shows that $\varphi s \in l_S(Ker(s))$. Then by assumption, we have $\varphi s \in Ss$. Hence $\varphi s = \hat{\varphi}s$ for some $\hat{\varphi} \in S$. To show that $\hat{\varphi}\iota = \varphi$. Let $s(m) \in s(M)$. Then $\hat{\varphi}\iota(s(m)) = \hat{\varphi}(\iota(s(m))) = \hat{\varphi}(s(m)) = \hat{\varphi}s(m) = \varphi s(m) = \varphi(s(m))$. Then M is quasi-small P-injective.

(2) $\Rightarrow$ (3) Let $s, t \in S$ with $s(M) \ll M$ and $Ker(s) \subset Ker(t)$. First we show that $l_S(Ker(t)) \subset l_S(Ker(s))$. Let $g \in l_S(Ker(t))$. Then $g(x) = 0$ for every $x \in Ker(t)$. To show that $g \in l_S(Ker(s))$, that is $g(x) = 0$ for every $x \in Ker(s)$. Let $x \in Ker(s)$. Since $Ker(s) \subset Ker(t)$, $x \in Ker(t)$. Hence $g(x) = 0$. Thus $g \in l_S(Ker(s))$. We now show that $St \subset l_S(Ker(t))$. Let $st \in St$ and let $x \in Ker(t)$. Then $t(x) = 0$, $st(x) = s(t(x)) = s(0) = 0$. Thus $st \in l_S(Ker(t))$. By (2), we have $St \subset l_S(Ker(t)) \subset l_S(Ker(s)) = Ss$. Then $St \subset Ss$.

(3) $\Rightarrow$ (4) Let $s, t \in S$ with $s(M) \ll M$. To show that $l_S(Ker(s) \cap Im(t)) = l_S(Im(t)) + Ss$. ($\subset$) Let $u \in l_S(Ker(s) \cap Im(t))$. Then $u(Ker(s) \cap Im(t)) = 0$. To show that $Ker(st) \subset Ker(ut)$. Let $x \in Ker(st)$. Then $st(x) = 0$, so that $t(x) \in Ker(s)$. We have $t(x) \in Im(t)$, hence $t(x) \in (Ker(s) \cap Im(t))$, so $ut(x) = 0$. Then $x \in Ker(ut)$. Since $st(M) \subset s(M)$, $st(M) \ll M$ by Proposition 2.2.3. Since $Ker(st) \subset Ker(ut)$ and $st(M) \ll M$, $Sut \subset Sst$ by (3). Since $ut = 1ut \in Sut \subset Sst$, $ut \in Sst$. Write $ut = vst$ for some $v \in S$. Then $ut - vst = 0$, so $(u - vs)t = 0$. Thus $(u - vs)t(x) = 0$ for all $x \in M$. Therefore $u - vs \in l_S(Im(t))$. It follows that $u = u - vs + vs \in l_S(Im(t)) + Ss$. ($\supset$) Let $u \in l_S(Im(t)) + Ss$. To show that $u \in l_S(Ker(s) \cap Im(t))$. That is $u(Ker(s) \cap Im(t)) = 0$, i.e., $ux = 0$ for every $x \in (Ker(s) \cap Im(t))$. Let $x \in Ker(s)$ and $x = t(m)$ for some $m \in M$. Since $u \in l_S(Im(t)) + Ss$, $u = v + \varphi s$ for some $v \in l_S(Im(t))$ and $\varphi \in S$. Thus $u(x) = v(x) + \varphi s(x) = v(t(m)) + \varphi(0) = 0 + 0 = 0$.

(4) $\Rightarrow$ (2) Let $s \in S = End_R(M)$ with $s(M) \ll M$. We have $1_M \in S$. Then by (4) we have $l_S(Ker(s) \cap 1(M)) = l_S(1(M)) + Ss$. Then $l_S(Ker(s)) = Ss$.
□

Let R be a Ring. A right R-module M is called *small principally injective* (briefly, *SP-injective*) [12] if, every R-homomorphism from a small and principal right ideal of R to M can be extended to an R-homomorphism from R to M. If R_R is an SP-injective, then we call R is a *right SP-injective ring*.

3.2.2 Corollary. *The following conditions are equivalent for a Ring R*:

(1) *R is SP-injective.*

(2) $lr(a) = Ra$ *for all* $a \in J(R)$.

(3) $r(a) \subset r(b)$, *where* $a \in J(R)$, $b \in R$ *implies* $Rb \subset Ra$.

(4) $l(r(a) \cap bR) = l(b) + Ra$ *for all* $a \in J(R)$, $b \in R$.

3.2.3 Proposition. *Let M be a principal module which is a self generator and let* $s = End(M)$. *If M is quasi-small P-injective, then* S *is a right SP-injective ring.*

Proof. To show that S is a right *SP*-injective ring. Let $s \in J(S)$ and let $\varphi : sS \to S$ be an S-homomorphism. Since M is a self generator, $Ker(s) = \sum_{t\in I} t(M)$ for some $I \subset S$. Since $s = s\cdot 1 \in sS$, $\varphi(s) = g$ for some $g \in S$. For any $t \in I$, we have $\varphi(s)t = gt$. Since $\varphi(s)t = \varphi(st) = \varphi(0) = 0$, $gt = 0$. Since $gt = 0$, $gt(M) = 0$ so $Im(t) \subset Ker(g)$. It follows that $Ker(s) \subset Ker(g)$. Then by Theorem 2.1.16, there exists an R-homomorphism $\alpha : s(M) \to M$ such that $\alpha s = g$. Since M is a principal module, by Proposition 2.9.5, $J(M) \ll M$. By Proposition 2.10.4, we have $J(S)M \subset J(M)$. By Proposition 2.2.3, $J(S)M \ll M$. Since $s \in J(S)$, $s(M) \ll M$. Since M is quasi-small P-injective, there exists an R-homomorphism $\hat{\alpha} : M \to M$ such that $\alpha = \hat{\alpha}\iota$ where $\iota : s(M) \to M$ is the inclusion map. Hence $\hat{\alpha}\iota s = \alpha s = g$. Define $\hat{\varphi} : S \to S$ by $\hat{\varphi}(f) = \hat{\alpha} f$ for every $f \in S$. Let $f_1, f_2 \in S$ such that $f_1 = f_2$. Then $\hat{\varphi}(f_1) = \hat{\alpha} f_1 = \hat{\alpha} f_2 = \hat{\varphi}(f_2)$. This shows that $\hat{\varphi}$ is well-defined. Let $f_1, f_2 \in S$ and $s \in S$. Then $\hat{\varphi}(f_1 s + f_2) = \hat{\alpha}(f_1 s + f_2) = \hat{\alpha}(f_1 s) + \hat{\alpha}(f_2) = \hat{\alpha}(f_1)s + \hat{\alpha}(f_2) = \hat{\varphi}(f_1 s) + (f_2)$. This shows that $\hat{\varphi}$ is an S-homomorphism. To show that $\varphi = \hat{\varphi}\iota$. Let $sa \in sS$. Then $\hat{\varphi}\iota(sa) = \hat{\varphi}(sa) = \hat{\alpha}(sa) = \alpha(sa) = (\alpha s)(a) = g(a) = (\varphi(s))(a) = \varphi(sa)$. This shows that $\hat{\varphi}$ is an extension of φ. □

3.2.4 Proposition. *Let M be a principal module which is a self generator. If M is quasi-small P-injective ,then*

(1) *If* $sS \oplus tS$ *and* $Ss \oplus St$ *are both direct,* $s, t \in J(S)$, *then* $l(s) + l(t) = S$.

(2) $lr(Ss) = Ss$ *for any* $s \in J(S)$.

Proof. (1) Define $\varphi : (s + t)S \to S$ by $\varphi(s + t)u = tu$ for every $u \in S$. If $(s + t)u = 0$, then $su = -tu \in sS \cap tS = 0$. Then $tu = 0$. Hence $\varphi(s + t)u = tu = 0$. This shows that φ is well-defined. Let $(s + t)u_1, (s + t)u_2 \in (s + t)S, v \in S$. Then $\varphi((s + t)u_1v + (s + t)u_2) = \varphi((s + t)(u_1v + u_2)) = t(u_1v + u_2) = tu_1v + tu_2 = \varphi((s + t)u_1)v + \varphi((s + t))u_2$. This shows that φ is an S-homomorphism. Since by Proposition 3.2.3, S is right SP-injective, there exists an S-homomorphism $\hat{\varphi} : S \to S$ such that $\varphi = \hat{\varphi}\iota$ where $\iota : (s + t)S \to S$ is the inclusion map. Hence $\hat{\varphi}(1)(s + t) = \hat{\varphi}(s + t) = \varphi(s + t) = t$, so $\hat{\varphi}(1)(s + t) = t$. Then $\hat{\varphi}(1)(s) + \hat{\varphi}(1)t = t$ and so $\hat{\varphi}(1)(s) = t - \hat{\varphi}(1)t = (1 - \hat{\varphi}(1))t \in Ss \cap St = 0$. Then $\hat{\varphi}(1)(s) = 0$ and $(1 - \hat{\varphi}(1))t = 0$. Hence $\hat{\varphi}(1) \in l(s)$ and $(1 - \hat{\varphi}(1)) \in l(t)$. Thus $1 = \hat{\varphi}(1) + (1 - \hat{\varphi}(1)) \in l(s) + l(t)$. Then $1 \in l(s) + l(t)$ so $l(s) + l(t) = S$.

(2) ($\supset$) Let $fs \in Ss$. To show that $fs \in l_S r_S(Ss)$. That is $fs(r(Ss)) = 0$, i.e., $fs(x) = 0$ for every $x \in r(Ss)$. Let $x \in r(Ss)$. Since $fs \in Ss$, $fs(x) = 0$. ($\subset$) Let $t \in lr(Ss)$. To show that $t \in Ss$. Define $\varphi : sS \to tS$ by $\varphi(su) = tu$ for every $u \in S$. Let $0 = su \in sS$. To show that $tu = 0$. That is to show that $tu(x) = 0$ for every $x \in M$. Let $x \in M$. Then $su(x) = 0$ so $tu(x) = 0$. This shows that φ is well-defined. Let $su_1, su_2 \in sS$ and $v \in S$. Then $\varphi(su_1v + su_2) = \varphi(s(u_1v + u_2)) = t(u_1v + u_2) = tu_1v + tu_2 = \varphi(su_1)v + \varphi(su_2)$. This shows that φ is an S-homomorphism. Since by Proposition 3.2.3, S is right SP-injective, there exists an S-homomorphism $\hat{\varphi} : S \to S$ such that $\iota_2\varphi = \hat{\varphi}\iota_1$ where $\iota_1 : sS \to S$ and $\iota_2 : tS \to S$ are the inclusion maps. We have $1 \in S$. Then $t = t\cdot 1 = \varphi(s\cdot 1) = \varphi(s) = \hat{\varphi}(s) = \hat{\varphi}(1)s \in Ss$. This shows that $lr(Ss) \subset Ss$. □

3.2.5 Proposition. *Let M be a quasi-small P-injective module and $s_i \in S$ with $s_i(M) \ll M$, $(1 \leq i \leq n)$.*

(1) *If $Ss_1 \oplus \ldots \oplus Ss_n$ is direct, then any R-homomorphism $\alpha : s_1(M) + \ldots + s_n(M) \to M$ has an extension in S.*

(2) *If $s_1(M) \oplus \ldots \oplus s_n(M)$ is direct, then $S(s_1 + \ldots + s_n) = Ss_1 + \ldots + Ss_n$.*

Proof. (1) Let $Ss_1 \oplus \ldots \oplus Ss_n$ is direct and let $\alpha : s_1(M) + \ldots + s_n(M) \to M$ be an R-homomorphism. Since M is quasi-small P-injective, for each i, $1 \leq i \leq n$, there exists an R-homomorphism $\varphi_i : M \to M$ such that $\alpha s_i(m) = \varphi_i s_i(m)$ for every $m \in M$. Since $s_i(M) \ll M$ for each $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} s_i(M) \ll M$ by Proposition 2.2.3(2), and we have $(\sum_{i=1}^{n} s_i)(M) \subset \sum_{i=1}^{n} s_i(M)$ which implies $(\sum_{i=1}^{n} s_i)(M) \ll M$ by Proposition 2.2.3(1). Since M is quasi-small P-injective, there exists an R-homomorphism $\varphi : M \to M$ such that, for any $m \in M$, $\varphi(\sum_{i=1}^{n} s_i)(m) = \alpha(\sum_{i=1}^{n} s_i)(m)$. To show that $\sum_{i=1}^{n} \varphi s_i = \sum_{i=1}^{n} \varphi_i s_i$. Let $m \in M$. Then $\sum_{i=1}^{n} \varphi_i s_i(m) = \varphi_1 s_1(m) + \varphi_2 s_2(m) + \ldots + \varphi_n s_n(m) = \alpha s_1(m) + \alpha s_2(m) + \ldots + \alpha s_n(m) = (\alpha s_1 + \alpha s_2 + \ldots + \alpha s_n)(m) = \alpha(s_1 + s_2 + \ldots + s_n)(m) = \alpha(\sum_{i=1}^{n} s_i)(m) = \varphi(\sum_{i=1}^{n} s_i)(m) = \varphi(s_1 + s_2 + \ldots + s_n)(m) = (\varphi s_1 + \varphi s_2 + \ldots + \varphi s_n)(m) = \varphi s_1(m) + \varphi s_2(m) + \ldots + \varphi s_n(m) = \sum_{i=1}^{n} \varphi s_i(m)$. This shows that $\sum_{i=1}^{n} \varphi s_i = \sum_{i=1}^{n} \varphi_i s_i$. Then $(\varphi_1 s_1 - \varphi s_1) + (\varphi_2 s_2 - \varphi s_2) + \ldots + (\varphi_n s_n - \varphi s_n) = 0$. Thus $(\varphi_1 - \varphi)s_1 + (\varphi_2 - \varphi)s_2 + \ldots + (\varphi_n - \varphi)s_n = 0$. Since $Ss_1 \oplus Ss_2 \oplus \ldots \oplus Ss_n$ is direct, $(\varphi_1 - \varphi) = (\varphi_2 - \varphi) = (\varphi_n - \varphi) = 0$. Then by Proposition 2.6.8, $(\varphi_1 - \varphi)s_1 = (\varphi_2 - \varphi)s_2 = \ldots = (\varphi_n - \varphi)s_n = 0$. Hence $(\varphi_i - \varphi)s_i = 0$, for all $1 \leq i \leq n$. Thus $\varphi_i s_i = \varphi s_i$, for all $1 \leq i \leq n$. To show that $\alpha = \varphi\iota$. Let $s_1(x_1) + s_2(x_2) + \ldots + s_n(x_n) \in s_1(M) + s_2(M) + \ldots + s_n(M)$. Then $\alpha(s_1(x_1) + s_2(x_2) + \ldots + s_n(x_n)) = \alpha s_1(x_1) + \alpha s_2(x_2) + \ldots + \alpha s_n(x_n) = \varphi_1 s_1(x_1) + \varphi_2 s_2(x_2) + \ldots + \varphi_n s_n(x_n) = \varphi s_1(x_1) + \varphi s_2(x_2) + \ldots + \varphi s_n(x_n) = \varphi(s_1(x_1) + s_2(x_2) + \ldots +$

$s_n(x_n)) = \varphi\iota(s_1(x_1) + s_2(x_2) + \ldots + s_n(x_n))$. Hence $\alpha(s_1(x_1) + s_2(x_2) + \ldots + s_n(x_n)) = \varphi\iota(s_1(x_1) + s_2(x_2) + \ldots + s_n(x_n))$. This shows that φ is an extension of α.

(2) ($\supset$) Let $\alpha_1 s_1 + \alpha_2 s_2 + \ldots + \alpha_n s_n \in Ss_1 + Ss_2 + \ldots + Ss_n$. To show that $\alpha_1 s_1 + \alpha_2 s_2 + \ldots + \alpha_n s_n \in S(s_1 + s_2 + \ldots + s_n)$. For each i, define $\varphi_i : (s_1 + s_2 + \ldots + s_n)(M) \longrightarrow M$ by $\varphi_i((s_1 + s_2 + \ldots + s_n)(m)) = s_i(m)$ for every $m \in M$. Let $0 = (s_1 + s_2 + \ldots + s_n)(m) \in (s_1 + s_2 + \ldots + s_n)(M)$. Then $s_1(m) + s_2(m) + \ldots + s_n(m) = (s_1 + s_2 + \ldots + s_n)(m) = 0$. Since $s_1(M) \oplus s_2(M) \oplus \ldots \oplus s_n(M)$ is direct, $s_1(m) = s_2(m) = \ldots = s_n(m) = 0$ so $s_i(m) = 0$. This shows that φ_i is well-defined. Let $(s_1 + s_2 + \ldots + s_n)(m_1), (s_1 + s_2 + \ldots + s_n)(m_2) \in (s_1 + s_2 + \ldots + s_n)(M)$ and $r \in R$. Then $\varphi_i((s_1 + s_2 + \ldots + s_n)(m_1)r + (s_1 + s_2 + \ldots + s_n)(m_2)) = \varphi_i((s_1 + s_2 + \ldots + s_n)(m_1 r + m_2)) = s_i(m_1 r + m_2) = s_i(m_1 r) + s_i(m_2) = s_i(m_1)r + s_i(m_2) = \varphi_i((s_1 + s_2 + \ldots + s_n)(m_1))r + \varphi_i((s_1 + s_2 + \ldots + s_n)(m_2))$. This shows that φ_i is an R-homomorphism. By the similar proof of (1) we have $(s_1 + s_2 + \ldots + s_n)(M) \ll M$. Since M is quasi-small P-injective, there exists an R-homomorphism $\hat{\varphi}_i : M \longrightarrow M$ such that $\varphi_i = \hat{\varphi}_i\iota$ where $\iota : (s_1 + s_2 + \ldots + s_n)(M) \longrightarrow M$ is the inclusion map. Then $s_i = \varphi_i(s_1 + s_2 + \ldots + s_n) = \hat{\varphi}_i(s_1 + s_2 + \ldots + s_n) \in S(s_1 + s_2 + \ldots + s_n)$. Hence $\alpha_i s_i = \alpha_i\hat{\varphi}_i(s_1 + s_2 + \ldots + s_n) \in S(s_1 + s_2 + \ldots + s_n)$ so $\alpha_1 s_1 + \alpha_2 s_2 + \ldots + \alpha_n s_n = \alpha_1\hat{\varphi}_1(s_1 + s_2 + \ldots + s_n) + \alpha_2\hat{\varphi}_2(s_1 + s_2 + \ldots + s_n) + \ldots + \alpha_n\hat{\varphi}_n(s_1 + s_2 + \ldots + s_n) = (\alpha_1\hat{\varphi}_1 + \alpha_2\hat{\varphi}_2 + \ldots + \alpha_n\hat{\varphi}_n)(s_1 + s_2 + \ldots + s_n) \in S(s_1 + s_2 + \ldots + s_n)$. ($\subset$) Let $\alpha(s_1 + s_2 + \ldots + s_n) \in S(s_1 + s_2 + \ldots + s_n)$. Then $\alpha(s_1 + s_2 + \ldots + s_n) = \alpha s_1 + \alpha s_2 + \ldots + \alpha s_n \in Ss_1 + \ldots + Ss_n$. □

3.2.6 Proposition. *Let M be a quasi-small P-injective module and $s_1(M) \oplus \ldots \oplus s_n(M)$ a direct sum of small and fully invariant M-cyclic submodules of M. Then for any fully invariant small submodule A of M, we have*

$$A \cap (s_1(M) \oplus \ldots \oplus s_n(M)) = (A \cap s_1(M)) \oplus \ldots \oplus (A \cap s_n(M)).$$

Proof. ($\supset$) Since $A \cap s_i(M) \subset A \cap (s_1(M) \oplus \ldots \oplus s_n(M))$ for each $i = 1, 2, \ldots, n$, we have $(A \cap s_1(M)) \oplus \ldots \oplus (A \cap s_n(M)) \subset A \cap (s_1(M) \oplus \ldots \oplus s_n(M))$. ($\subset$) Let $a = \sum_{i=1}^{n} s_i(m_i) \in A \cap (s_1(M) \oplus \ldots \oplus s_n(M))$. To show that $\sum_{i=1}^{n} s_i(m_i) \in (A \cap s_1(M)) \oplus \ldots \oplus (A \cap s_n(M))$. Let $\pi_k : \bigoplus_{i=1}^{n} s_i(M) \to s_k(M)$ be the projection map. Since for each i, $(1 \leq i \leq n)$, $s_i(M)$ is small and fully invariant, by Proposition 2.1.17, $Ss_i(M) \subset s_i(M)$. Thus $\bigoplus_{i=1}^{n} Ss_i(M)$ is direct, so $\bigoplus_{i=1}^{n} Ss_i$ is direct. By Proposition 3.2.5, π_k has an extension $\hat{\pi}_k : M \to s_k(M)$ such that $\pi_k = \hat{\pi}_k \iota$ where $\iota : s_1(M) \oplus s_2(M) \oplus \ldots \oplus s_n(M) \to M$ is the inclusion map. Let $m_i \in M$. Then $s_i(m_i) = \pi_i(\sum_{i=1}^{n} s_i(m_i)) = \hat{\pi}_i \iota(\sum_{i=1}^{n} s_i(m_i)) = \hat{\pi}_i(\sum_{i=1}^{n} s_i(m_i)) = \hat{\pi}_i(a) \in A \cap s_i(M)$. Hence $\sum_{i=1}^{n} s_i(m_i) = s_1(m_1) + s_2(m_2) + \ldots + s_n(m_n) \in A \cap s_1(M) \oplus A \cap s_2(M) \oplus \ldots \oplus A \cap s_n(M)$.
□

3.2.7 Theorem. *Let M be a quasi-small P-injective module, $s, t \in S$ and $s(M) \ll M$.*

(1) *If $s(M)$ embeds in $t(M)$, then Ss is an image of St.*

(2) *If $t(M)$ is an image of $s(M)$, then St embeds in Ss.*

(3) *If $s(M) \cong t(M)$, then $Ss \cong St$.*

Proof. (1) Let $f : s(M) \to t(M)$ be an R-monomorphism. Since M is quasi-small P-injective, there exists an R-homomorphism $\hat{f} : M \to M$ such that $\iota_2 f = \hat{f}\iota_1$ where $\iota_1 : s(M) \to M$ and $\iota_2 : t(M) \to M$ are the inclusion maps. Define $\sigma : St \to Ss$ by $\sigma(ut) = u\hat{f}s$ for every $u \in S$. Let $0 = ut \in St$. To show that $Im(\hat{f}s) \subset Im(t)$. Let $\hat{f}s(m) \in \hat{f}s(M)$. Then $\hat{f}s(m) = fs(m) \in t(M)$. To show that $\sigma(ut) = 0$, i.e., $u\hat{f}s(m) = 0$ for every $m \in M$. Let $m \in M$. Then $u\hat{f}s(m) = ufs(m) = ut(y)$ for some $y \in M$. Hence $u\hat{f}s(m) = ut(y) = 0$. This shows that σ is well-defined. To show that σ is a left S-homomorphism.

Let $u_1(t)$, $u_2(t) \in St$ and $v \in S$. Then $\sigma(vu_1t + u_2t) = \sigma((vu_1 + u_2)t) = (vu_1 + u_2)\hat{f}s = vu_1\hat{f}s + u_2\hat{f}s = v(u_1\hat{f}s) + u_2\hat{f}s = v\sigma(u_1t) + \sigma(u_2t)$. To show that σ is an S-epimorphism. Let $ks \in Ss$. To show that $Ker(\hat{f}s) \subset Ker(s)$. Let $x \in Ker(\hat{f}s)$. Then $\hat{f}s(x) = 0$, so $fs(x) = \hat{f}s(x) = 0$. Since f is monic, $s(x) = 0$. Then $x \in Ker(s)$. Since $s(M) \ll M$ and $\hat{f} : M \to M$ is an R-homomorphism, $\hat{f}s(M) \ll M$ by Proposition 2.2.4. Since M is quasi-small P-injective, $Ss \subset S\hat{f}s$ by Lemma 3.2.1. Then $s = 1 \cdot s = u\hat{f}s$ for some $u \in S$. Hence there exists $kut \in St$ such that $ks = \sigma(kut)$.

(2) Let $f : s(M) \to t(M)$ be an R-epimorphism. Since M is quasi-small P-injective, there exists an R-homomorphism $\hat{f} : M \to M$ such that $\iota_2 f = \hat{f}\iota_1$ where $\iota_1 : s(M) \to M$ and $\iota_2 : t(M) \to M$ are the inclusion maps. Define $\sigma : St \to Ss$ by $\sigma(ut) = u\hat{f}s$ for every $u \in S$. It is clear that σ is a left S-homomorphism. Let $ut \in Ker(\sigma)$. Then $0 = \sigma(ut) = u\hat{f}s = ufs$. To show that $ut = 0$, i.e., $ut(m) = 0$, for all $m \in M$. Let $m \in M$. Since f is an R-epimorphism, $f(s(a)) = t(m)$ for some $a \in M$. Then $ut(m) = uf(s(a)) = 0$.

(3) Follows from (1) and (2). □

Lists of References

[1] F. W. Anderson and K. R. Fuller, "**Rings and Categories of Modules**," Graduate Texts in Math. No.13 ,Springer-verlag, New York, 1992.

[2] V. Camillo, "Commutative Rings whose Principal Ideals are Annihilators," **Portugal Math.**, Vol 46, 1989. pp 33-37.

[3] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, "**Extending Modules**," Pitman, London, 1994.

[4] A. Facchini, "**Module Theory**," Birkhauser Verlag, Basel, Boston, Berlin,1998.

[5] T.Y. Lam, "**A First Course in Noncommutative Rings**," Graduate Texts in Mathematics Vol 131, Springer-Verlag, New York, 1991.

[6] S. H. Mohamed and B. J. Muller, "**Continuous and Discrete Modules**," London Math. Soc. Lecture Note Series 14, Cambridge Univ. Press, 1990.

[7] W. K. Nicholson and M. F. Yousif, "Principally Injective Rings," **J. Algebra**, Vol 174, 1995. pp 77-93.

[8] W. K. Nicholson and M. F. Yousif, "Mininjective Rings," **J. Algebra**, Vol 187, 1997. pp 548-578.

[9] W. K. Nicholson, J. K. Park and M. F. Yousif, "Principally Quasi-injective Modules," **Comm. Algebra**, 27:4(1999). pp 1683-1693.

[10] N. V. Sanh, K. P. Shum, S. Dhompongsa and S.Wongwai, "On Quasi-principally Injective Modules," **Algebra Coll**.6: 3, 1999. pp 269-276.

[11] L. Shen and J. Shen, "Small Injective Rings," arXiv: Math., RA/0505445 vol 1, 2005.

[12] L.V. Thuyet, and T.C.Quynh, "On Small Injective Rings, Simple-injective and Quasi-Frobenius Rings," **Acta Math. Univ. Comenianae,** Vol 78(2), 2009. pp 161-172.

[13] R. Wisbauer, "**Foundations of Module and Ring Theory**," Gordon and Breach Science Publisher, 1991.

[14] P.B. Bhattacharya, S.K. Jain and S.R. Nagpaul, "**Basic Abstract Algebra**," The Press Syndicate of the University of Cambridge, second edition, 1995.

Lists of References (Continued)

[15] B. Hartley and T. O. Hawkes, "**Ring, Modules and Linear Algebra**," University Press, Cambridge, 1983.

[16] S. Wongwai, "On the Endomorphism Ring of a Semi-injective Module," **Acta Math.Univ. Comenianae**, Vol 71, 2002. pp 27-33.

[17] S. Wongwai, "Almost Quasi-mininjective Modules," **Chamjuri Journal of Mathematics**, Vol 2, 2010, no. 1. pp 73-79.

[18] S. Wongwai, "Small Principally Quasi-injective Modules," **Int. J. Contemp. Math. Sciences**, Vol 6, 2011, no. 11. pp 527-534.

[19] S. Wongwai, "Quasi-small P-injective Modules," **Journal of Science and Technology RMUTT**, Vol 1, 2011. no. 1. pp 59-65.

[20] Friedrich Kasch and Adolf Mader, "Rings, Modules and the Total," **Birkhauser Verlag**, Basel, Switzerland, 2004.

[21] P. Yordsorn and S. Wongwai, "A note on quasi–small *P*–injective Modules," **Proceeding of The 5th Conference on Fixed Point Theory an Applications,** July 8-9, 2011, Lampang , Thailand. pp 117.

The 5th Conference on Fixed Point Theory an Applications at

Lampang Rajabhat university

July 8-9, 2011

Appendix

Conference Proceeding

Paper Title “A note on quasi–small *P*–injective Modules”

The 5th Conference on Fixed Point Theory an Applications

At Lampang Rajabhat university

July 8-9, 2011

The 5th Annual Conference on

Fixed Point Theory an Applications

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Abstracts

In celebration of the 40th anniversary of Lampang Rajabhat University

Content

2B10 : 9 July 2011
Time : 11.10 - 11.30

THE 5[th] CONFERENCE ON FIXED POINT THEORY AND APPLICATIONS

Faculty of Science, Lampang Rajabhat University July 8-9, 2011

A NOTE ON QUASI-SMALL P-INJECTIVE MODULES

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Let M be a right R−module. A right R-module N is called *M-small principally injective* (briefly, *M-small P-injective*) if, every R-homomorphism from an M-cyclic small submodule of M to N can be extended to an R-homomorphism from M to N. In this paper we give some characterizations and properties of quasi-small principally injective modules.

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